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ABSTRACT

The approach to teaching group theory used in this report was to give students many different kinds of examples of the concept being taught. The functions of models and examples in terms of student preferences and the psychology of the learning experience are discussed. Six representations of the tetrahedral group were developed as test materials. One was chosen for the initial learning presentation and then related to the other models through a series of transfer tests. Comparisons are made between data obtained on different tests on the same students, as well as between similar data obtained from different subjects. The procedure at the conclusion in terms of its potential value to both the classroom and research is examined. Included in the report is an examination of the related literature. (MP)

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A COMPARATIVE STUDY OF THE LEARNING
AND UNDERSTANDING OF A MATHEMATICAL
GROUP STRUCTURE IN DIFFERENT
REPRESENTATIONS.

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ABSTRACT

After an appraisal of the functions of representations in the learning of mathematics the understanding of different representations of the same mathematical structure is discussed in terms of their psychological equivalence. This equivalence is held to manifest itself in performances resembling those of the bilingual subject, who speaks both languages in isolation and translates from each into the other. Since a testee may succeed in relating the representations of some samples of a mathematical structure but fail to relate others, measurement becomes possible on a number scale. At each point of this scale preferences may be revealed for one of the representations or one direction of the translation. Consideration of these preferences leads to other types of measurement.

The related literature is examined for its concern with mathematical representations in general and with the effects of changes in these representations in particular.

Six representations of the tetrahedral group are developed as test materials. One of them is chosen for presentation by itself in an initial learning session and then related to the other representations in a series of transfer tests. Comparisons are made between different numerical data obtained from the same subjects, and between the same data obtained from different subjects. A series of hypotheses are formulated, and in conclusion

the whole procedure is examined for its potential value to the classroom and to research.

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The stimulation resulting from this contact enabled the author to develop the investigations reported here after he departed from Sherbrooke to take up an appointment in Halifax, Nova Scotia. There the Education Authorities co-operated generously and gave permission for his tests to be run in their schools. Some further experimentation was subsequently made possible by the Integrated School Board in St. John's, Newfoundland, where the author also had gratuitously placed at his disposal the use of an electronic computer by Memorial University.

During the preparation of the present report the author was greatly encouraged by the critical suggestions of Mr. John Williams, Director of the LVCE Research Institute in London, England. It was through his willingness to communicate across the Atlantic that the work could be brought to its conclusion. The typing of the final version by Mrs. Margaret Howard is much appreciated.

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INTRODUCTION

Teaching involves communication, and communication is always dependent on some medium such as words, pictures or physical objects. In the course of the present century educators have focused more and more of their attention on the needs of the learner. Their attempts to provide him with the right sort of environment have therefore led to the search for improvements in the medium of communication as well.

In the teaching of language this concern has expressed itself in the approach to the written word by carefully graded stages, sometimes with the use of the Initial Teaching Alphabet as an intervening representation. At elementary levels the difficulties for the pupil are here mainly created by the complexity of the rules that govern spelling and the correct formation of sentences. The content of the language is mostly one with which children are already acquainted from experiences they have gained casually outside of school.

In the teaching of mathematics the symbolisms of algebra and arithmetic are less complex in their rules, but the content that is expressed in them makes greater demands on the understanding than the content of ordinary language. Mathematical statements express relationships between entities which are themselves first or higher order abstractions from reality. The symbolisms by which they are communicated encapsule the relevant information concisely

for the professional, but often perplex the learner to the point where he misunderstands or fails to understand altogether. One possible approach to helping him over his difficulties is then to present the content to him in one or several different forms.

In keeping with their general commitment to the needs of the learner those responsible for mathematics education have also sought to make school mathematics more readily intelligible through the modification or total abandonment of traditional approaches. But the need for promoting the understanding of the subject by young children through the provision of a concrete and stimulating environment only found wide acceptance after World War II. It was then that a large variety of manufactured teaching aids reached the classroom, among them the structural materials of Stern and Cuisenaire whose use was carefully directed by their authors in new and comprehensive mathematics courses for the Junior School. In the late fifties the theme was taken up by Dienes and others who saw the key to satisfactory mathematical abstraction in the use of several representations in conjunction. His innovations were publicised at a time when many reformers called for the infusion of new mathematical content into the curriculum as well. Since content and its representations can be varied independently of one another the two types of reform could be readily combined, with the result that the teacher of today can select from a wide range of mathematical topics and is offered an

equally wide choice of instructional materials to help him in their presentation.

On the theoretical side the learning of mathematics as structure has been looked upon by Dienes, Bruner and others as an extension of concept learning, involving the development of strategies and the generation of hypotheses on the part of the learner as he moves towards the solution of a problem as his goal. In part these strategies are determined by the store of cognitive experiences he brings to bear on the learning situation, in part by the kind of instruction he is given, and in part by the perceptual stimuli that he has available for use to assist his thinking. According to Piaget operating with conventional symbolic systems which have no concrete referents is characteristic of the mature adult. The young pupil stands at the opposite end of the scale and must progress towards this internalisation by stages through suitably directed rapport with his environment. Since representations are part of this environment their selection becomes as critical an issue as the selection of the content of the mathematics curriculum itself. Representations must be matched to the pupil's developmental level and replaced when a change in this level occurs; this amounts to conceiving of different representations as a chain that extends from the initiating representation of the first lesson to those representations which form part of the equipment of the trained mathematician later on.

Given the importance of representations in mathematics learning it would seem essential to accumulate knowledge about them individually as well as about the ways in which they interrelate. A broad classification of representations into concrete, iconic and symbolic has been provided by Bruner, but both within and between each of these categories there remains much room for further distinctions. Research has tended to focus on the effectiveness of different representations, especially the representations provided by the structural materials of Dienes, Stern and Cuisenaire, in promoting the understanding of arithmetic in its conventional symbolic representation. Although this emphasis may be justified by the central position of arithmetic in the curriculum, it should not cause the possibility of variations in the form of enquiry as well as in the mathematical content and its representations to become overlooked. If the range of content and representations now at the disposal of the teacher at every stage of the curriculum is wider than ever before, then it would seem only natural to match these provisions by a diversification of the procedures that have been and can be evolved for their evaluation. At which stage of the pupil's development and for the teaching of which content is a particular set of instructional materials most effective? This would seem an obvious and very general type of question for research to attempt to answer, if only because with increased enrichment of the curriculum and with greater abundance of the means by which it may be unfolded the need for systematic

orientation and the avoidance of wasteful teaching strategies becomes correspondingly greater.

The work reported in the present Thesis was undertaken for the purpose of studying the pupil's transition between pairs of representations in detail. Some guidance for the procedures evolved was obtained from two sources, bilingualism and cross-modal perception. Bilingualism manifests itself in three abilities, the ability to speak in either of two languages alone, the ability to translate information from one language into the other, and the ability to reverse the direction of this translation. Cross-modal perception involves the recognition of two sensory stimulations such as those of sight and touch as emanating from one and the same object. It manifests itself in the ability to identify the same object after it has been sensed in each mode independently. The concept of equivalence has been used to account for cross-modal perception. Since there is a close and obvious analogy between this equivalence and the impartiality of the fluent bilingual to his two languages the concept of the psychological equivalence of two representations was here defined as being testable through those performances by which the three abilities of the bilingual are distinguished. During the measurement of this equivalence the possibility always exists that the testee will reveal preferences for one representation or one direction of the translation. Further concepts and measurements must therefore be defined to allow for the lack of equivalence also to be taken into account.

Although the process of passing from one mathematical representation to another does not as a rule resemble the process of translation between two languages very closely, this translation was incorporated into the test procedures so as to ensure that in its measurement no aspect of the concept of psychological equivalence was being overlooked. It turned out that measurements arising from this translation lowered the overall assessment of psychological equivalence considerably. More information about the subjects was obtained as a result, yet despite their increased complexity the tests could be kept within reasonable dimensions. The opportunity of adapting the whole approach to other mathematical structures or to investigations outside of mathematics and of obtaining a satisfactory assessment of psychological equivalence and its associated concepts within no more than two classroom periods is offered to any investigator who might be interested in pursuing the subject further. Indeed, it would be through the resulting diversification that the generality of the hypotheses which have here been advanced on the strength of only limited observation could be put more seriously to the test.

CHAPTER 1

REPRESENTATIONS AND THEIR PSYCHOLOGICAL EQUIVALENCE

General considerations

A representation is any part of the environment which we interpret as standing for something other than itself. The rope represents safety to the climber, execution to the hangman. Neither may know or be interested in knowing that in itself it is just plaited fibre. The object arouses their anticipations as each recalls instances of the context in which he has seen it used.

It is through our ability to create and understand representations that our experiences become organized, enabling our adaptive behaviour to grow in the direction of ever-increasing complexity. The use of representations appears in the make-believe activities of the young child and extends to the mature creations of science, language and the arts. The width of this range is reflected in the kind of environmental material that serves as representation, also in the degrees of organisation that this material has to undergo before it can serve its purpose. The empty box is turned into the representation of a house within a few seconds in the playroom, but the grammar of a written language takes centuries to evolve. The first representation is make-shift and private to the child, the second has been perfected to the point where it serves a

large number of people to hold fast and pass on the products of their culture. Once the rules governing its use have been understood a given language may be manipulated so as to express a large range of new creations. In the same way each special symbolism of mathematics and science serves progress for a considerable time before the expression of some and highly specialised content makes it imperative that it be supplemented or replaced.

If representations are subservient to their content they assume this subordinate role only for those who have mastered their use. Education provides the paths to the attainment of this mastery; for this reason representations must always be more than one of its minor concerns. The gap between the box in the playroom and the textbook in the library is there to be bridged, and increased insight into the details of the bridging process can help the educator to promote the transition.

Progressive growth of representational competence.

An extensive study of how the use of representations emerges during the child's development has been made by Piaget and summarised by Flavell.¹ At the sensory-motor stage he

is capable only of linking, one by one, the successive actions or perceptual states with which it gets involved. Representational thought, on the other hand, through its symbolic capacity has the potential for simultaneously grasping, in a single, internal epitome, a whole

¹ Flavell's excerpts which follow are based on Piaget (1941), (1950), (1951), (1954a) and (1954b).

sweep of separate events. It is a much faster and more mobile device which can recall the past, represent the present, and anticipate the future in one temporally brief, organised act . . .

By its very potential to get outside the immediate present representational thought can eventually extend its scope well beyond actual, concrete acts of the subject and actual, concrete objects in the environment. The ultimate in this freedom from concrete reality is the symbolic manipulation (as in scientific and mathematical thought) of entities which are not even picturable, let alone tangible. ¹

Even at the sensory-motor stage the infant is capable of treating a perceptual cue as a sign that some event is soon to follow. Through imitation he acquires his first signifiers

capable of internally representing for him the absent significate. With the growth and refinement of his capacity to imitate the child is eventually able to make internal imitations as well as external, visible ones. He is able to evoke in thought, as opposed to actually carrying out in reality, imitations made in the past. ²

Piaget distinguishes between two kinds of signifier.

There are

signs whose meanings are socially shared and which are arbitrary in the sense that they bear no systematic resemblance to their significates. Words are the commonest signs, but there are also others (e.g. mathematical and scientific symbols).

1 Flavell(1963)pp. 151-152.

2 Ibid., pp. 152-153.

Symbols, on the other hand, are more or less private, noncodified signifiers which usually do bear some physical similarity to their referents.

The very first signifiers are not linguistic signs but things like the piece of cloth which Jacqueline used to represent a pillow in pretended going-to-sleep actions. It is not the acquisition of language which gives rise to the symbolic function. Quite to the contrary, the symbolic function is a very general and basic acquisition which makes possible the acquisition of both private symbols and social signs. Language is the vehicle par excellence of symbolization, without which thought could never become really socialised and thereby logical. But thought is nonetheless far from being a purely verbal affair, neither in its fully formed state nor, above all, in its developmental origins. In essence, what happens is that language, first acquired through the auspices of a symbolic function which has arisen earlier, will reflexively lend tremendous assistance to the subsequent development of the latter.

The first signifiers, then have the properties of private symbols rather than of social signs. The first words the child acquires function at most as semisigns, signs which are thoroughly impregnated with private symbol-characteristics. These semisigns refer much more to idiosyncratic and ever-changing schemas of action than to fixed and stable classes of objective realities. Just as the piece of cloth signifies a past schema of going-to-sleep-on-pillow, so may an early word like "mommy" signify (among other things, depending upon context), not a class of objects, but simply that a child wants something. The fact of the matter is that for a long time the child finds verbal signs as such both difficult to grasp and generally unsuitable for representational expression of the entities with

which he is preoccupied. He responds to this state of affairs both by continuing to rely heavily on nonverbal symbols and by assimilating words to his symbolic orientation, i.e. treating the word as just another private symbol.¹

A threefold classification of representations

Piaget's analysis lays stress on the fact that in the early stages the child's representational materials are concrete and three-dimensional, and that his use of spoken language is tentative and far removed from that of the adult. It is in the use of language with its complex rule system that much adaptation lies ahead. This adaptation is also supported by use of a third representation - the picture,² in which one dimension has been lost but some geometric correspondences with the three-dimensional world remain preserved. The theme has been taken up by Bruner in his classification of representations into concrete (or enactive),³ iconic and symbolic:

1 Flavell(1963),pp. 154 - 155.

2 Cf. Piaget(1951),p. 78 and p.280.

3 The term 'enactive' would seem to be more appropriate when the representation involves movement of the whole body rather than just the manipulation of concrete materials. Such movements are well described in Dienes (1973). Under both conditions the representation is multi-sensory, but in the enactive case kinaesthetic sensation is added to the sensations of sight and touch.

Concrete representations are sometimes referred to as 'embodiments'. See Bruner(1971),p. 66.

What is abidingly interesting about the nature of intellectual development is that it seems to run the course of these three systems of representation until the human being is able to command all three . . .

Take our first and second bench marks, the question of representation: how the child gets free of present stimuli and conserves past experience in a model, and the rules that govern storage of information from this model. Let me suggest that there are probably three ways in which human beings accomplish this feat. The first is through action. We know many things for which we have no imagery and no words, and they are very hard to teach to anybody by the use of either words or diagrams or pictures. If you have tried to coach somebody at tennis or to teach a child to ride a bike, you will have been struck by the diagrammatic impotence of the teaching process.... There is a second system of representation that depends upon visual or other sensory organization and upon the use of summarizing images.... We have come to talk about the first form of representation as enactive, the second as iconic. Iconic representation is principally governed by principles of perceptual organization and by the economical transformations in perceptual organization.... Finally, there is representation in words or language. Its hallmark is that it is symbolic in nature.... Symbols (words) are arbitrary (as Hockett puts it, there is no analogy between the symbol and the thing, so that "whale" can stand for a very big creature and micro-organism for a very small one), they are remote in reference,¹ and they are almost always highly productive or generative in the sense that a language or any symbol

1 Bruner's use of the word 'symbol' is in keeping with common usage. In Piaget's account above it is the 'sign' that is remote in reference.

system has rules for the formation and transformation of sentences that can turn reality over on its beam ends beyond what is possible through actions or images.¹

Bruner's threefold classification has provided a theoretical background for the commercial production of three-dimensional teaching materials and specially illustrated texts, with the result that the exclusive use of adult language and symbolisms has disappeared from many classrooms.

Although the utility of the classification is beyond dispute its conceptual foundations stand in need of some clarification if misunderstanding is to be avoided. The opening paragraph of the present chapter stressed the distinction between a representational material and the context in which it is placed. Bruner's classification seems to rest in part on the material as such and in part on its context. If the material is considered in isolation then only a two-fold classification seems possible. Concrete materials are three-dimensional and perceptible² by both sight and touch, whereas pictures and written words are two-dimensional and perceptible only by sight. If, on the other hand, the material is considered in a definite context then the property of being remote in reference can belong to concrete materials and pictures no less than to symbols. The figure of the knight in chess or the card of the jack of hearts in bridge, for example, are both symbols because their geometric properties in no way suggest their function

1 Bruner (1971), pp. 10 - 12.

2 Such materials are sometimes referred to as multi-sensory.

in these games. Their elaborate design merely serves the purpose of attracting and holding the player's attention while he is pre-occupied with the strategies of his game. Likewise it is possible for content to be suggested by written characters which commonly serve as pure symbols. The four groups 00, 01, 10 and 11, for example, can be used to refer to four objects which differ from each other in one or two respects, and in ordinary language content is suggested by the onomatopoeic word.

A way out of this dilemma is to say that the word or mathematical symbol is usually remote in reference and that pictures and three-dimensional materials frequently are not. This is in keeping with the way children commonly use these materials, also with the general use of models and diagrams as concrete and iconic representations. Moreover, if the materials of a given kit have been designed to serve as a specific representation the classification tends to become transferred to the materials themselves. Because of these general trends, however, the possibility that multipurpose materials such as the Dienes Logic Blocks ¹ may be used symbolically ² should not be overlooked. In the last resort the nature of every

1 See E.S.A.(1976). These Blocks consist of the 4 shapes of triangle, circle, square and rectangle. Each shape appears in 3 colours, 2 sizes and 2 thicknesses. This makes $4 \times 3 \times 2 \times 2 = 48$ pieces in all.

2 Suppose the first 9 numbers are represented by Logic Blocks in the following order: red triangle, blue triangle, yellow triangle, red square, blue square, yellow square, red circle, blue circle, yellow circle. The shape then symbolises the quotient and the colour the remainder obtained when any of the 9 numbers is divided by 3.

representation depends on the stipulations that are made at the time of its construction. In the case of concrete or iconic representations these stipulations make use of the physical properties of the materials involved to help suggest the content to some degree or even display it in its entirety. But unless the pupil is the inventor of a representation himself he cannot be sure whether or how the physical properties of a particular material that engage his attention will serve as a representation. A hoop placed on the floor might represent a circle, a closed set, or the empty set; or it might just mark off a portion of the floor on which he is expected to work.

Further considerations of the three-fold classification

The potential of symbolic representations exceeds that of the other two in most respects. Where models are used in the teaching of anatomy or geography the instruction is to some extent supported by words, but not every course of verbal instruction employs models or diagrams. For the purposes of filing permanent records the two-dimensional drawings or photograph rather than the three-dimensional model interlaces well with verbal accounts since both representations fit on to the two-dimensional paper.

The use of models and drawings for the purposes of displaying information is limited to situations in which such information can be associated with points in space. Such association is not usually possible in ethics or literature but becomes readily available in many areas of mathematics and science. Where information concerning a

geometry of not more than three dimensions is to be represented the association is immediate and can make full use of the physical properties of the representational material. With a rise in the number of dimensions the association can no longer be made to cover all dimensions simultaneously.

Often iconic representations in the form of drawings provide an immediate overview of the content where the corresponding verbal description would be lengthy. The ordinary road map is an obvious case in point. Should the information be of the kind that requires theoretical justification then the diagram with its summarizing powers still assists in focusing on the centre of the discussion, but the logical analysis leading to the justification will generally be performed by means of words and symbols.

Skemp¹ has shown how the theorem that angles in the semi-circle are always right angles can be proved by means of a sequence of diagrams animated by arrows to suggest the direction of the reasoning (Fig. 1 - 1).

1 Skemp (1971), p.105.

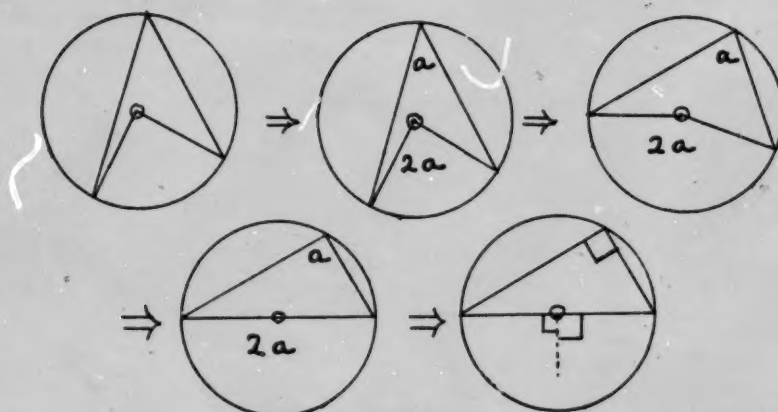


Figure 1 - 1

Animated diagram showing the logical steps in Skemp's proof.

The last of the five circles summarises the result, and this together with the remaining four or other circles would normally be interlaced with a verbal and symbolic form of the same proof.

Even the summarizing powers of the diagram are limited by the amount of detail that it is possible to represent. A Venn diagram will easily show all pairwise intersections of three or even four sets. After that the reader finds it progressively more difficult to disentangle the regions, and a sufficiently high number of sets is soon reached for which the use of this representation would totally fail in its purpose. Again, the entire set of real numbers between 0 and 1 is quickly represented by a line segment. On this segment it is then possible to

make subdivisions corresponding to numbers such as $\frac{1}{2}$, $\frac{5}{6}$ or $\frac{1}{12}$ with a degree of accuracy that is sufficient for many practical purposes. But no matter how good the drawing instruments may be, the discrimination between numbers can never approach that which a symbolic representation makes possible.

Structure

In general there is no clear-cut answer to the question: "What does a representation represent?" A religious sect may use the public representation of language and unfold its beliefs to others at great length. When efforts are made to understand what has been said they culminate in a maze of possible interpretations as an acceptable conceptual framework is sought for holding these beliefs together.

In many segments of human discourse, on the other hand, content is definable without ambiguity, and for the expression of such content the symbolisms of mathematics and logic are well suited. A small number of basic concepts such as 'velocity', 'atom' or 'number' are defined at the outset and the further development of the subject matter is then always describable in terms of these.

Basic to all branches of mathematics is the concept of 'set'. This concept constitutes a second order abstraction from the properties which characterise the

1 With the use of the decimal representation it is possible to distinguish between any two real numbers in principle. In practice the distinction must end after a finite number of decimal places, but this number can be made arbitrarily large.

the simple concepts formed by the child during his cognitive development. Simple concepts such as 'dog' or 'house' are known as soon as the child has abstracted properties which certain animals and buildings have in common. He is then able to use these words correctly¹ whenever he encounters animals or buildings possessing these properties. This is the first stage of abstraction. The second stage consists in recognising something that 'dog' and 'house' have in common, namely that they are sets or classes of things with common properties. The concept 'set' thus comprises all such particular sets whose individuality may be stressed by referring to them as the 'set of dogs' and 'the set of houses'.

Mathematics concerns itself with sets such as the set of real numbers or the set of points in the plane, in its more abstract branches also with sets of elements which are not further specified because it is immaterial whether they are numbers or points. Delimiting the range of the discussion by describing or listing a set is a necessary start,² but at the same time not an achievement from which alone mathematics derives its stature. The dynamics of the subject as an ever-growing conceptual

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- 1 As long as the child can only use 'dog' or 'house' to refer to his own house these words do not represent fully formed concepts to him.
 - 2 Mathematicians have used the concept of a set explicitly only in fairly recent times. But its presence was implicitly acknowledged during extensions such as the extension of the real to the complex numbers.

organisation rests on the systematic exploration of the relationships between elements of a given set or indeed of different sets. Through these relationships elements and groups of elements become individualised as parts within an organic whole¹, and when these profounder aspects are taken into consideration the mathematical set becomes a mathematical structure².

The implied analogy with the structure of a building holds in essence, since the parts and their interconnections are identifiable without ambiguity in both cases. Only the procedures that make this identification possible are different. Whereas the building is directly open to inspection, the parts of the mathematical structure can only

1 See Dienes and Jeeves (1965), p.12.

2 A formalised theory of structural learning has been proposed by Scandura (1973). In this the behavioural characteristics exhibited by subjects who know rules play a central role. After equating the terms 'rule', 'schema' and 'principle' he writes (p.12): "A rule can be denoted by a function whose domain is a set of stimuli and whose range is a set of responses. The concept and the association become special cases. A concept can be represented by a function in which each stimulus is paired with a common response, while an association can be viewed as a function whose defining set consists of a single stimulus-response pair." He exemplifies this function (p.21) with an example from number conservation: "Each cardinal number (e.g. 1, 2, 3,.....) may be defined as the class of all sets which can be put into one-to-one correspondence with one another. A child who has learned to conserve (cardinal) numbers, then, has learned a rule for determining whether or not it is possible to pair in one-to-one fashion the objects in two (or more) arbitrary collections." The approach thus offers a rationale for rejecting earlier stimulus-response theory in accounting for mathematics learning, and it also formulates clearly the difference between structural learning and the learning of simple concepts.

be exposed through the intervening medium of a representation. To be functional this representation must cater to complexity and size. Where a small finite structure is involved a representation is easily invented and explained, but even the most cursory glance at the history of mathematics testifies to the arduous route which brought the representations of infinite structures such as the counting or the rational numbers to their present form.

Some representations of mathematics in public use

1. Cardinal number

Small cardinal numbers ¹ are represented by children or primitive peoples by means of a tally made on the fingers or with marks on a wall. The tally is iconic in that it displays the quantity that has been counted. As soon as this quantity rises to any size the reading of the tally becomes time consuming. All numeration systems have sought to overcome the difficulty by some form of grouping and the representation of this grouping by a new mark of different appearance from the first. At this point the representation becomes, in part at least, symbolic, since the total number of marks used in the grouped tally no longer displays the quantity. As step-by-step counting continues the choice between continuing

1 The set of cardinal numbers is here understood to be (0, 1, 2, 3,.....). From this set only the subset of counting numbers (1, 2, 3,.....) is representable by the tally and numeration systems which are now obsolete.

with the tally of existing symbols or further grouping with new symbols presents itself again and again. Since there is no limit to the growth of number even a seemingly economical grouping by twenties or sixties rather than tens still demands the use of a large number of symbols. In our Hindu-Arabic system of numeration this difficulty is overcome by effecting the symbolisation not through the symbol alone, but through the symbol-in-position. As a consequence the tally has to disappear altogether since two marks representing a tally of 'two' would occupy two different positions. A new symbol is introduced to represent the number 'two' at a stage where it might not have seemed necessary. Because of the symbolic value now accorded to position the number of distinct symbols required can be kept to ten, and only the number of positions has to grow to whatever size the represented number requires. But positions, unlike symbols, do not have to be invented or remembered. They merely have to be counted off from right to left, starting with the unit position.

This highly functional representation, through which the common operations of arithmetic have become fairly simple to execute, has superseded all other representations which sought to retain some form of tally. Being a representation of structure it must also be capable of reflecting the structure's dynamism. For this purpose the ten number symbols in position are supplemented by symbols for addition and other operations, also by symbols for inequality, equality and bracketing. The use of each

symbol and position is controlled by a system of rules which leaves us free to focus on whatever parts of the structure of cardinal number happen to be of interest. It is in fact the carefully planned constraint of the rules that provides this freedom, just as the rules of grammar allow the user of language to give his thoughts the direction he intends.

Even here the size of a number can make the notation cumbersome. The representation of the number

7 9 8 6 8 9 5 4 3 2 5 7 9

extends over 13 positions. Commonly one is not interested in more than the first 3 or 4 figures. When rounded off to the first 4 figures the number reads

7 9 8 7 0 0 0 0 0 0 0 0 0

There are 9 zeros following the second 7 and for the tally of these 9 zeros another shorthand is provided. By using this 9 as superscript to the right of the 10 we can shorten the representation of this number to

7 9 8 7 x 10¹²

and for an even larger number of zeros the economy provided by the new notation would be greater still.

2. Rational number

In the representation of rational numbers such as 'two fifths' there is a choice between using integer pairs in the fractional representation and between the decimal representation which grows quite naturally out of the

representation of cardinal number from the use of position value to the right of the unit position. Both representations are in common use. They have a common range over which they can conveniently be used as alternatives and also separate individual ranges in which only one of them is functional. The calculation 'one half plus one fifth' is easily performed in both representations. The decimal representation serves much better in applied contexts involving metric measurement or in theoretical discussions about the difference between rational and irrational numbers¹. On the other hand, in a calculation such as 'one third times one seventh' the fractional representation assumes the simple form

$$1/3 \times 1/7 = 1/21$$

whereas the use of the decimal representation would involve the multiplication of the two infinite series 0.33333... and 0.142857.... . Choosing the latter representation might lead to a challenging academic exercise but could never be considered as an alternative in practice.

3. Simultaneous equations

In the solution of not more than two simultaneous linear equations one can represent each equation by a line drawn on graph paper and look for the co-ordinates of the point of intersection of these lines (Fig. 1 - 2).

1 Every rational number expressed in fractional form will, on division of the numerator by the denominator yield a decimal which is either terminating or recurring. The decimal representation then leads us to define decimals which are neither terminating nor recurring. Such decimals are called non-rational or irrational.

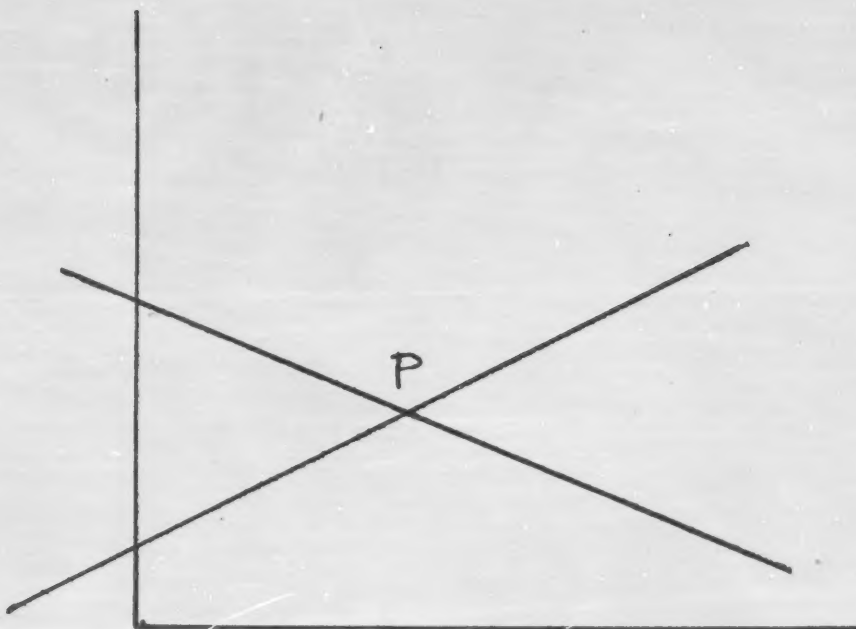


Figure 1 - 2

The graphical solution of two simultaneous equations by means of two lines intersecting at the point P.

Alternatively one can solve the equations by the purely algebraic methods of either elimination or substitution. The symbols commonly used for representing the two unknowns are x and y , sometimes other letters of the alphabet. As long as the solutions are integers ¹ there is little to choose between the representations. The graphical method requires a grasp of the elementary techniques of plotting

¹ Rational solutions extending over more than 2 places of decimals would be difficult to obtain even from a very accurate drawing.

straight lines and the algebraic methods require an understanding of some elementary manipulations.

Suppose that we now wish to introduce the two degenerate cases of (i) no finite solution and (ii) and infinite number of finite solutions. As an illustration of case (i) we could consider

$$x + y = 6 \text{ together with } 2x + 2y = 7$$

and as an illustration of case (ii) we could consider

$$x + y = 6 \text{ together with } 2x + 2y = 12.$$

The graphical representations of the two cases look like this :

Case (i)

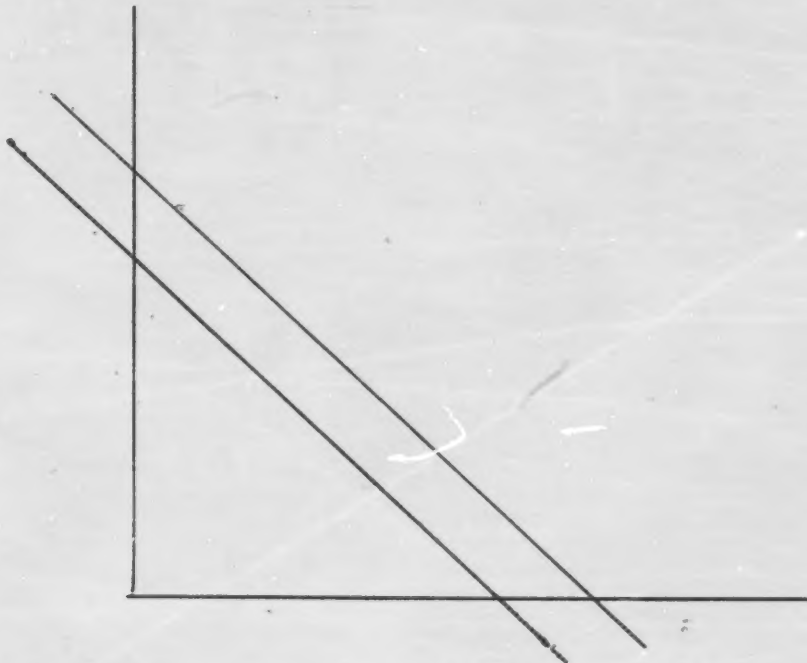


Figure 1 - 3.

When there is no finite solution the two lines are parallel.

Case (ii)

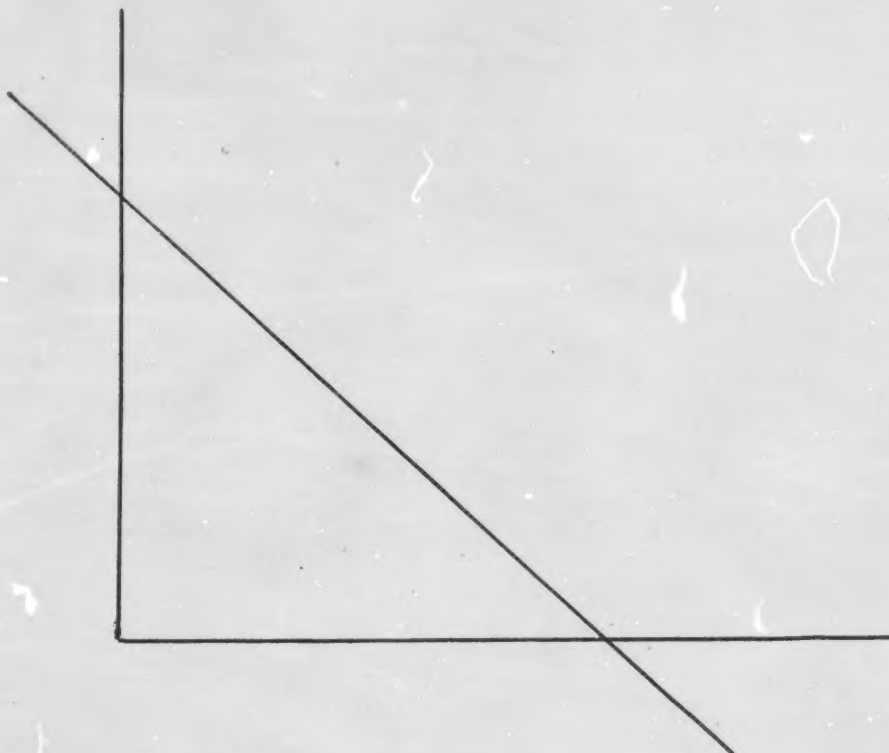


Figure 1 - 4.

Since both equations represent the same line only one line can be drawn.

The graphical representations suggest the degeneracy of the two cases as soon as the lines are drawn. For case (i) they are parallel and for case (ii) they are coincident. Algebraic methods on the other hand, lead to the equation $0 = 5$ in case (i) and to the equation $0 = 0$ in case (ii). Both results look perplexing whereas a meaningful interpretation of the degeneracy is directly obtainable from the graphs.

If the number of simultaneous equations is increased from 2 to 3 the geometric representation becomes considerably

more involved. It is now necessary to construct a three-dimensional model of three planes, or to sketch these places on two-dimensional paper in a convincing way. A solution in integers can be read off from an accurate three-dimensional model, but not from a sketch on two-dimensional paper. The algebraic method gains in importance as a result. But we saw a moment ago that this method does not suggest a meaningful interpretation of the degenerate cases. Consequently it would seem appropriate to strengthen the understanding through the use of the two methods in conjunction, interpreting each of the equations as a plane in three space, inspecting the constants of the equations to see whether any pair of planes is parallel or coincident, and using algebra to find the solutions once it is known that they exist. A point will be reached where the three-dimensional model or sketch of the planes need no longer be present. The interpretation they suggest is then fully conceptualized but remains useful in understanding the degenerate cases.

Later still it becomes necessary to consider simultaneous equations in an arbitrary number of unknowns. This transition from a fixed number 3 to an arbitrary number n is an instance of generalisation¹ and typical of the way in which mathematical systems are developed so as to increase in power and scope. As far as the two representations are concerned, the limitations of the graphical method and the superiority of the algebraic method now become

1 See Dienes (1961) for a detailed analysis of generalisation.

strikingly apparent. It is not possible to construct physical models of n -dimensional space, with the result that the geometrical entities ¹ representing the equations in this space must be conceptualised right from the start. The only assistance that a graphical representation can provide has to be derived from the earlier models. As far as the algebraic method is concerned there is no change of principle during this generalisation, though naturally the procedure for finding the solution is lengthier and more elaborate. Also, it may happen that the number of equations will exceed the number of letters in the alphabet. To overcome this problem it is then usual to employ just a single letter x and distinguish the unknowns by adding suffixes ² to it. This will then give the infinite set of symbols $x_1, x_2, x_3, \dots$ in place of only the finite set provided by letters of the alphabet.

For contexts where reference is to be made to a system of simultaneous equations without explicit mention of their number mathematicians have developed a representation known as matrix algebra.³ In this representation a system of an arbitrary number of simultaneous equations is referred to by one single equation

$$\underline{Ax} = \underline{b}$$

-
- 1 These entities are referred to as hyperplanes, flat subspaces of 1 dimension less than n , the number of dimensions of the space.
 - 2 For difficulties experienced with the suffixed notation see the summary of Thorndike's paper in Chapter 2, pp. 71 - 73.
 - 3 The details are to be found in any introductory textbooks on the subject.

and if it is non-degenerate its solution may formally¹ be expressed as

$$\underline{x} = A^{-1}\underline{b}$$

Some didactic representations in the classroom

From the last section it will be apparent that as mathematical structures grow in size and generality their representations assume symbolic form and become stabilised in a system of rigid rules. Without this they would not be functional. But in classrooms where mathematical thinking is fostered only in its embryonic forms the choice of representations is far less restricted. Small structures containing around 50 elements or less are now easily represented by concrete materials or pictures, and the same possibility exists for selected samples of infinite structures such as the cardinal or rational numbers. Considerations of ultimate functionality give way to considerations of the didactic value a particular representation has for a particular group of learners. A representation which encourages them to explore and understand structural relationships is preferable to one which expresses these relationships in a concise but possibly incomprehensible manner.

1. Difference relations

With the aid of the Dienes Logic Blocks² it is possible to represent the difference relations existing in

1 Finding actual solutions in a particular case would of course entail analysing the content of the symbols $\underline{x}$, A and b in detail.

2 E.S.A.(1976).

in a set of 4 elements in a variety ¹ of ways as shown in Figures 1 - 5 to 1 - 8. It is supposed that each arrangement of the Blocks lies on a sheet of cardboard on which lines may be drawn. Differences are differences in colour, ² size, or both colour and size. The number of lines relating any pair of Blocks to each other is equal to the number of differences between them.

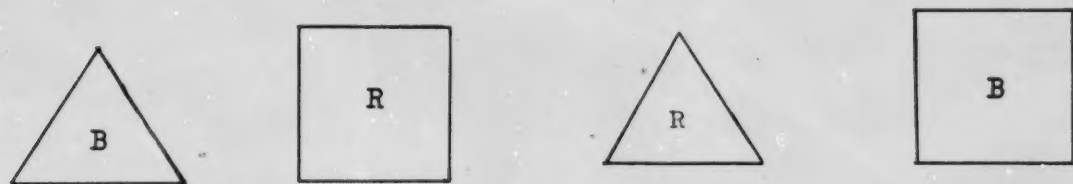


Figure 1 - 5

All the difference relations are left implicit.

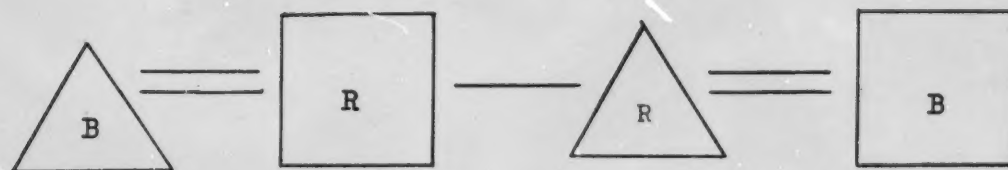


Figure 1 - 6

Difference relations between adjacent Blocks are displayed by an appropriate number of lines.

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- 1 Some displays of difference relations will be found in Seaborne (1974).
 - 2 The colours blue and red are here denoted by the capital letters B and R.

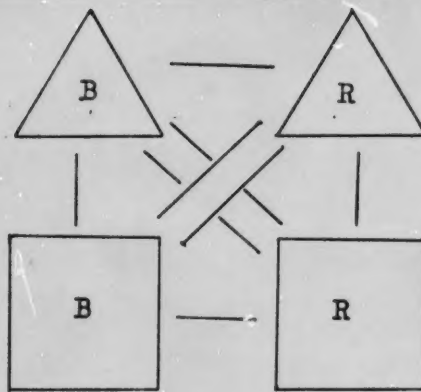


Figure 1 - 7

The 6 difference relations between all possible pairs of Blocks are displayed in a systematic manner.

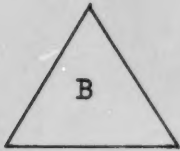
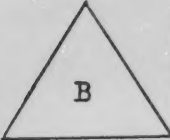
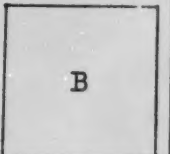
				
		—	—	==
	—		==	—
	—	==		—
	==	—	—	

Figure 1 - 8

A tabular arrangement showing also that the difference between each Block and itself is zero. (The duplication of each Block on the outside of the table is for representational purposes only and does not imply a duplication of the number of elements.)

In Figure 1 - 5 all the difference relations are left implicit. The display is of the kind that would be used to familiarise children with the Blocks and make them think about the ways in which they differ. Their first attempt to represent some of the differences explicitly might then result in the display of Figure 1 - 6. The arrangement of the Blocks in Figure 1 - 7 is superior to the previous one in that every Block is next to all the other three. This ensures that every difference will be considered. Finally the tabular arrangement of Figure 1 - 8 is more sophisticated in that it also provides explicit information about the difference between any Block and itself and duplicates each Block for representational purposes only.

In Figure 1 - 5 the rules of the representation amount to nothing more than stipulations as to how the properties possessed by the four Blocks are to be interpreted in relation to each other. In Figure 1 - 6 these rules are augmented by rules governing the use of single and double lines, and in the remaining two Figures rules governing the exact positioning of the Blocks are also added. One form of variation in the representation is thus provided by changes in these rules. Another form could be provided by changes in the Blocks themselves. Instead of two different colours and two different shapes it would be possible to represent the same structure of difference relations with two thicknesses and two sizes. Such changes would be changes in the stimulus material of the representation and could be made independently of changes in the rules which state how this material is to be interpreted and organised.

2. Cardinal number

A representation for the teaching of cardinal number is provided by the Dienes Multibase Arithmetic Materials.¹ These consist of sets of wooden blocks designed to represent the grouping of the unit in various number bases. In the rectangular version the unit is represented by a small cube known as a 'single'. The first grouping of the unit is represented by a 'stick' equivalent in length to a preselected number ² of singles. The second grouping is represented by a square known as a 'flat'. This square is formed by grouping sticks according to the same number. Finally there is a third grouping of the 'flats' known as a 'cube'. In base three, for example, the single and the three groupings look like this:

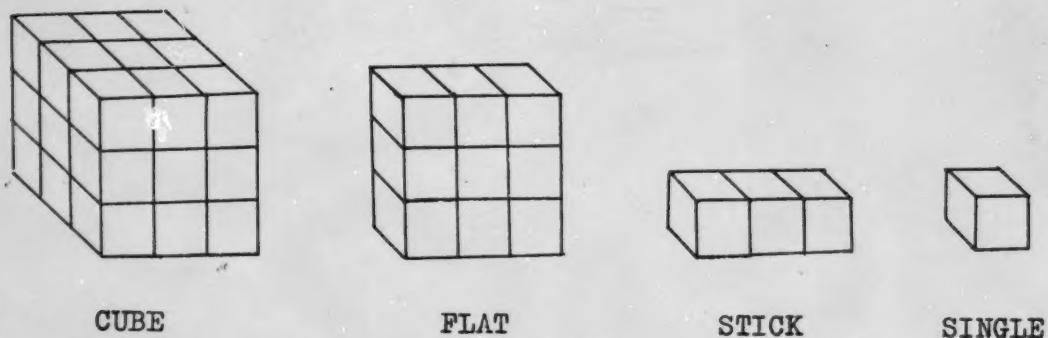


Figure 1 - 9

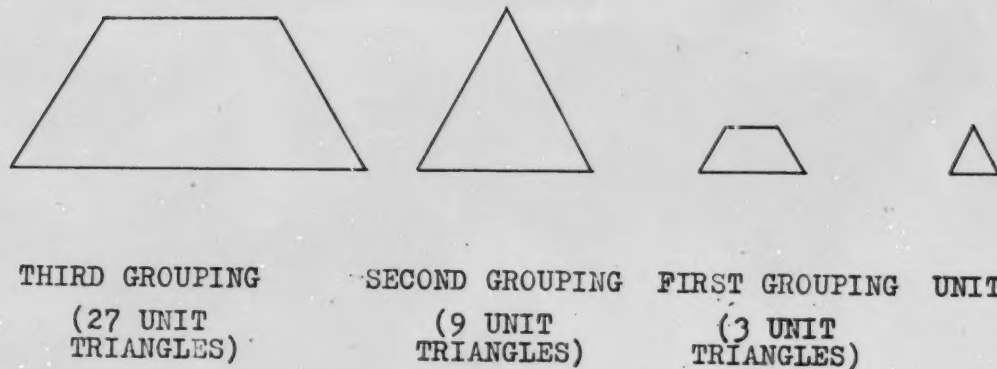
The rectangular version of the Multibase Arithmetic Materials for base three.

1 See E.S.A. (1976).

2 This number is the number of the base.

Each piece representing a grouping is notched, with the result that the quantity represented is directly ascertainable by inspection. In the early stages of teaching there is thus no need to insist on the particular positioning of the pieces as shown here. But since the kit is intended eventually to lead to a grasp of the representation of cardinal number by symbols-in-position the appropriate placing of the pieces has to occur at some stage.

In the triangular version the same principle of grouping according to a preselected number is employed. This time, however, the pieces are not notched and the third grouping extends the piece of the second grouping in two dimensions instead of three. In base three the unit and the three groupings look like this:



The triangular version of the Multibase Arithmetic Materials for base three.

Arithmetic Materials serve to teach the understanding of cardinal number and also of its conventional symbolic representation.¹ They are all similar in that the rules according to which their stimulus material is organised derive from one and the same principle - the choice of a number for the base. If this number is held constant then the pieces of the triangular and rectangular versions are in one : one correspondence with one another and can be exchanged by simple substitution. By way of contrast the representations provided by the structural materials of Stern and Cuisenaire² help the understanding of cardinal number but not of its symbolic representation. The rules for the organisation of their stimulus material rest on the association of number with length and bear no relation to those of the Multibase Arithmetic Materials. Nevertheless, within the common range over which all these representations are functional any property of the structure of cardinal number can be communicated by any of them equally well.

Additionally it is often possible within one and the same rule system of a representation to introduce another type of variation, namely a variation of the path by which the answer to the same question is reached. In any of the representations of cardinal number that have been discussed it is, for example, possible to have pupils work out the sum $2 \times (3 + 5)$ either as $(2 \times 3) + (2 \times 5) = 6 + 10 = 16$ or as $2 \times 8 = 16$. Similarly, in the case of the algebraic

1 See p.37 supra.

2 See Appendices S2 - 2 and S2 - 3.

representations of simultaneous equations pupils can be asked to use either the method of elimination or the method of substitution. Alternative paths of this kind are made possible by the laws of the structure, and can therefore be taken in any representation which gives these laws expression ¹. As a system of rules and stimuli the representation remains unchanged, and in essence the content also remains the same. But since a change in the path involves the selection of different stimuli and rules from the system ² it is appropriate to speak of a change in representation here as well.

Equivalence

From this account of some well-known representations it is apparent that situations in which the same content is expressed in two different representations arise from mathematics in its own right as well as from pedagogical considerations. Whenever two representations are suited for transmitting the same information they can be termed equivalent ³. To be equivalent they must have been correctly constructed, but within the framework of this stipulation there remains much freedom in the selection of the stimulus material and the rules which direct its organisation. Consequently any comparison of two equivalent

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- 1 The change from any one algebraic representation to the graphical representation would constitute a change in the path together with a change in the stimulus material and rule system of the representation. The procedures intervening between the problem and its solution would then have nothing in common.
 - 2 In the case of one and the same language too it is generally possible to express the content of a sentence or paragraph in a variety of ways.
 - 3 Scandura (1973, p.13) speaks of equivalence classes of representations. Instances of such classes are the equivalence classes of fractions, with each class representing one and the same rational number.

representations may reveal (i) similarities in the stimulus material, (ii) overlap in the stimulus material, (iii) similarity in the rules, or (iv) identity in the rules up to substitution of the stimulus material that is organised.

Between the graphical and algebraic representations of simultaneous equations none of these relationships are detectable. Similarity in the stimulus material can range from strict geometric similarity between large and pocket-size versions of the same blocks to one or more prominent common features such as colour or dimensionality¹.

Overlap in the stimulus material exists between the rectangular versions of the Multibase Arithmetic Materials for all the number bases, since each version employs the same² piece for the unit (single). Similarity in the rules is to be found between these same versions. In fact, if the base number is referred to as 'n', then one set of rules couched in general terms will serve all the bases, provided that it is correctly interpreted for each base. Identity in the rules up to substitution exists between the rectangular and the triangular versions for any given base. One : one correspondences³ are easily set up between the pieces of each version and the rules for manipulating corresponding pieces are always the same.

1 See p. 29 supra for the distinction between the concrete and other representations in terms of dimensionality.

2 In the same way the '0' and the '1' occur in the symbolic representation of every number base.

3 Such correspondences also exist between the words of the same language when expressed in conventional letters and in morse. The Initial Teaching Alphabet was introduced to ensure their existence between the spoken and the written word.

Since the equivalence of representations depends on the stipulations creating each representation all the above relationships play only an incidental part. The understanding and co-ordination of two representations, on the other hand, may be promoted or impeded by them.

Psychological equivalence

When two representations are equivalent as defined in the last section it is possible to pass from one to the other as in the translation of languages. We shall say that for any person who can work in them equally well and translate in both directions with equal success they are psychologically equivalent. Equivalence is thus a necessary condition for psychological equivalence.

The term 'equivalence' has been used in connection with the perception of one and the same object in two different sensory modes such as those of sight and touch. An obvious analogy between this type of perception and the perception of representations suggests itself and may be supported by the correspondences listed in Table 1 - 1. Both types of perception involve learning and interpretation. But whereas in the case of the physical object learning can result from contact between the object and the perceiving subject alone, the correct interpretation of the representation of a mathematical structure involves the prior acceptance of some arbitrary stipulations. In this type of perception learning must therefore be specially directed ¹.

1 Cf. Gibson (1967), pp. 419 - 420.

At the present time the psychological processes by which the matching of two representations is mediated in both types of perception are far from being clearly understood. In the case of the physical object arguments centre on whether specific experiences in each sensory mode are necessary and have to be linked by association later before a satisfactory matching can occur, or whether some sensory stimulation carries information that is invariant¹ over several modes, so that even people born blind can learn from touch what discrimination² by sight would be like.

In the case of the perception of a mathematical structure the content is invariant over the representations, but there is nothing corresponding to the unity of the senses which has been postulated for the other type of perception. Knowing the content possibly enables one to invent a new representation but does not enable one to predict the rules and nature of the stimulus material of a new representation that has been created by someone else.

1 Gibson (1967), pp. 215 - 231.

2 Ibid., p.216.

Table 1 - 1

Some correspondences in the analogy between the perception of a physical object in two different sensory modes and the understanding of a mathematical structure in two different representations.

<u>Physical Object</u>	<u>Mathematical Structure</u>
1. Learning to interpret the stimuli received from the object in each sensory mode. 2. Identifying all the parts of an object in each sensory mode. 3. Matching the stimuli from each part of the object that is sampled.	1. Learning to interpret the stimulus material and the rules of each representation. 2. Identifying all the parts of a mathematical structure in each representation. 3. Matching the representation of each part of the structure that is sampled.

However, the introduction of didactic representations into the classroom rests on the principle that prior knowledge of the content gained through the didactic representation will make a conventional representation easier to understand. To the extent that the two representations differ

in their rules and stimulus material the transfer is thus mediated through the common content alone. Often it is also mediated through similarity in the rules, and if the rules are identical it may be mediated through correspondences in the stimulus material alone¹.

In the area of bilingualism too the process of translation seems as yet to be poorly understood in its details.² On one view³ there are two types of bilingualism developing from the indirect and direct methods of language teaching respectively. In the indirect method the same meanings are thought to mediate responses in either language, whereas in the direct method each language has its own set of meanings developing in isolation and made to overlap with each other during subsequent translation. According to Kolers the method by which the second language is learned has no effect on bilingual performance once a certain level of skill is reached. This would point to the ultimate

1 See p. 54 supra. If the rules that define the use of the stimulus material are identical it is possible to by-pass the content altogether in translation. A telegraphist can translate or rather transliterate a sentence into morse even if he does not understand its meaning. Under these circumstances any assessment of psychological equivalence must be based on the correct use of each representation independently. Sometimes also identity in the stimulus material may be mistaken for identity in content. It is not uncommon to hear pupils refer to the symbol pair '10' as 'ten' even when its use is governed by the rules of a base other than base ten.

2 Vetter (1969), p.93.

3 Ervin and Osgood (1954).

formation of some optimal internal organisation¹ through which fluent transcoding between two languages becomes possible. In the testing of this hypothesis, however, it was found² that subjects still had a tendency to produce different associations for the same word in the two languages. This may have been due to the absence of any tight control of the content during the initial learning of the languages and during the periods where they were used independently. If two languages are learned in different countries or if one language is used at school and the other in the home it seems highly unlikely that identity of content will be strictly preserved, but the learning of a new mathematical structure in two previously unfamiliar representations may be quite a different matter.

That transitions between mathematical representations occur in some form or other as part of the regular curriculum is apparent from what has been said about the representations of rational numbers and simultaneous equations.³ Since during such transitions the essentials of the structure become distinguished from the accidental features of each representation the psychological importance of passing between didactic and conventional representations has led to the formulation of the principle of perceptual variability by Dienes.⁴ It is held that the employment of

1 Kolers (1963), p. 299.

2 Ibid., pp 291 - 300.

3 See pp.39-46supra.

4 Dienes (1963), p. 158.

this principle as part of teaching strategy will reduce the risk of misunderstanding by the pupil and leave him better prepared for changes in representation that occur in the natural course of events.¹

Principles of measurement

Since the present definition of psychological equivalence is in keeping with the attainments revealed by the professional translator of languages these attainments can also guide the development of its measurement. They consist of the expression of content in each representation separately together with the translation² of given content from either language into the other. Four components in the manifestation of psychological equivalence become distinguishable as a result.

In order to measure psychological equivalence a sample of a subject's performance containing all four components must be taken. Psychological equivalence is demonstrated to the extent that performance is equally distributed over the components. If it is unevenly distributed then the subject is

1 Dienes (1963), p. 159. See also the paper of Thorndike discussed in Chapter 2, pp. 71 - 73.

2 Translation is mediated through the understanding of the common content. A test of translation, however, differs from a transfer test of the conventional type in that it does not measure the degree to which knowledge of one representation is facilitated by knowledge of the other. Both representations are used simultaneously and no performance by subjects who know only one representation is measured as a control.

experiencing stress. There are three types of stress according as discrepancy in performance arises between the expression of content when each representation is used independently, or between the translation of content from one representation into the other in either direction. Finally there is the possibility that performance may be lacking totally or to the same degree in all four components. In that event the subject is equally ignorant¹ of both representations. To characterise performance completely it is necessary to provide measures of psychological equivalence, equally distributed ignorance² and stress. The details of this provision are fully described in the last part³ of Chapter 4.

Summary

After a general discussion of representations in relation to the learner's development the implications of some taxonomy in common use were examined. A brief consideration of the concept of a mathematical structure then led to the detailed examination of some conventional and didactic representations in mathematics. The distinction between their stimulus material and the rules of its

-
- 1 If there is no positive performance at all then there is no way of telling without further investigation whether the failure has resulted from ignorance of the content or the representation. In the case of partial failure the two causes can be distinguished, if provision is made to test knowledge of the content in each representation before translation. Such provision is conspicuously absent from conventional tests of translation between languages.
 - 2 Equally distributed ignorance may be calculated from the other two measures and the sample size.
 - 3 pp.125 - 135.

organisation was elaborated. The equivalence and the psychological equivalence of two representations were defined, and the principles governing the measurement of psychological equivalence and its associated concepts were discussed in outline.

CHAPTER 2

RELATED LITERATURE

As might be expected, the expansion in the volume of instructional materials for the classroom after World War II has been accompanied by substantial theoretical discussions¹ about their function and purpose as well as a large number of published reports and unpublished dissertations offering designs for their evaluation.

The most common approach of the evaluative studies is to assess the effectiveness of a set of instructional materials by one or more criterion tests in which questions are put to subjects in either or both of the symbolisms of arithmetic and ordinary language. The experimental group which has enjoyed the use of the materials is then compared on its performance in these tests with a control group. This makes for a transfer design of the conventional type but does not tell us anything definite about the psychological equivalence of the conventional representation and the representation provided by the instructional materials under consideration. Frequently the contents of the mathematics to which the experimental and control groups are exposed are not exactly matched, and sometimes also divergences in the teaching strategies of the two groups make the two treatments overlapping rather than coincident.²

1 See, for example, the two Yearbooks of the National Council of Teachers of Mathematics (N.C.T.M. 1943 & 1973).

2 Williams (1965), p. 37.

Even with perfect coincidence the results will only lead to the conclusion that the experimental group was helped, hindered or left unaffected by the additional representation. They may have understood both representations, but all details of discrepancies between their performance in each of them remain undisclosed.

These considerations take the core of the present chapter away from the main stream of research in the evaluation of representations and leave us with only a small number of studies in which the investigators have strictly controlled the content and taken measurements of the performance in each representation separately. However, inasmuch as the factors that make for psychological equivalence are potentially present whenever subjects work in two or more representations, brief reference has also been made to some important studies that are of only marginal relevance. In keeping with the need to maintain a satisfactory distinction between the peripheral and core literature, the chapter has been divided into two sections.

Peripheral Literature

Two samples of evaluative studies on concrete representations have been tabulated by Fennema (1975) and are reproduced in Tables 2 - 1 and 2 - 2. Both tables indicate that the effects of such representations are greater with younger¹ children, an observation which is also endorsed by the Canadian Council for Research in Education (1964).

¹ See Appendix S2 - 1 on p. 213 for the approximate correspondences between grades and age levels in North American Schools.

Table 2 - 1

A Summary of some Studies comparing the Cuisenaire approach with traditional approaches.

Author	Grade level	Test used	Significant difference in favour of Cuisenaire or traditional materials	Mathematical content
Aurich (1963)	First	Standardized achievement	Cuisenaire treatment	Total range of first-grade work.
Hollis (1964)	First	Standardized achievement	Cuisenaire treatment	Total range of first-grade work.
Crowder (1965)	First	Standardized achievement	Cuisenaire treatment	Total range of first-grade work.
Nasca (1966)	Second	Standardized achievement Cuisenaire achievement	Neither treatment Cuisenaire treatment	Total range of second-grade work.
Passy (1963)	Third	Standardized achievement	Traditional treatment	Computation and arithmetic reasoning.
Haynes (1963)	Third	Author constructed Standardized achievement	Neither treatment Neither treatment	Multiplication
Lucow (1963)	Third	Author constructed	Neither treatment	Multiplication and division.

Table 2 - 2

A Summary of some Studies comparing approaches using
concrete materials other than Cuisenaire with traditional approaches.

Author	Grade level	Concrete model	Test used	Significant difference in favour of:	Mathematical content
Lucas (1966)	First	Dienes Attribute Blocks	Standardized achievement and author constructed	Dienes treatment for conservation of number and conceptualization of mathematical principles; traditional for computation and solving of verbal problems	Identified in Piagetian terms; multiplication of relations and addition-subtraction relations
Ekman (1966)	Third	Counters	Author constructed	Neither treatment at end of instructional period; concrete-model group on a retention test	Addition and subtraction algorithms
Dawson & Ruddell (1955)	Fourth	Many diverse models	Author constructed	Concrete-model group	Division of whole numbers
Norman (1955)	Third	Concrete and semi-concrete models (number lines, drawings and counters)	Author constructed	Neither treatment at the end of instruction; concrete and semiconcrete-model group at the end of two weeks	Division of whole numbers

Table 2 - 2 continued

Author	Grade level	Concrete model	Test used	Significant difference in favour of:	Mathematical content
Howard (1950)	Fifth & Sixth	Concrete and semi-concrete models	No information available	Neither treatment at the end of instruction; concrete-model group three months later	No information available
Mott (1959)	Fifth & Sixth	Many multi-sensory aids	Standardized achievement	Neither treatment	Measurement
Spross (1962)	Fifth & Sixth	Concrete aids that had cultural significance	Standardized achievement	Neither treatment	Total range of fifth to sixth grade work
Price (1950)	Fifth & Sixth	Multisensory aids	No information available	Neither treatment	Division of fractions
Anderson (1957)	Eighth	Various visual-tactual devices	Author constructed	Neither treatment	Area, volume, & Pythagorean theorem

Only the study of Passy (1963) gives a decisive verdict against the concrete representation.

In England a two-year study was conducted on first and second year junior children by the National Foundation for Educational Research in the school years 1961 - 1963. The elaborate design by which the Cuisenaire and Dienes approaches to arithmetic were compared with each other and with the traditional approach has been fully explained by Williams (1972). The quantitative comparison, based on the results of three cognitive tests and an attitude questionnaire, was four-dimensional. After the first year the Cuisenaire experimental children were superior in all dimensions except attitude, and the Dienes experimental children inferior to the control group in all dimensions. After two years, however, this situation changed markedly. The Cuisenaire children surpassed their controls in all dimensions. The Dienes children also surpassed their controls in all dimensions except attitude and were about equal to the Cuisenaire children on the three cognitive tests. From this broadly based comparison one can only guess at possible reasons behind the time delay before the Dienes treatment became effective. The children in this treatment took much longer to adjust, but this could have been due to the whole approach surrounding the Dienes materials rather than to differences which were strictly differences in representation. There is nothing in the Cuisenaire or Stern ¹ approaches that corresponds to the

¹ See Appendices S2 - 2 and S2 - 3.

use of different number bases. Since questions in the criterion test inclined towards single or double digit numbers only, the Dienes approach may have amounted to a substantial digression from activities which could lead to success in these tests more directly; the children involved in the Dienes approach would therefore have required additional time before they could adjust to this digression and equal the Cuisenaire and the control children in other respects as well.

Multi-dimensional comparison is also to be found in the results collected by Brownell (1964) over 3 years from about 1400 third year junior school children in England and Scotland by tests and personal interview. An important component in his assessment was conceptual maturity as revealed in the ability of the children to give reasons for their mathematical manipulations. Brownell concluded that the quality of the instruction seemed more important¹ in the promotion of this maturity than the choice of a conventional, Cuisenaire or Dienes approach. The effects of the approaches and of the special representations employed in them were insufficiently pronounced to give a decisive verdict in favour of any one of them, but unfortunately in investigations of this type the quality of instruction is a variable whose control is beset with many practical difficulties².

1 Brownell (1964), p.309.

2 Biggs (1967), p.98.

A survey of Biggs (1967) extending over 4000 children in their 3rd junior year in 87 schools in England assessed subjects on verbal reasoning, mechanical arithmetic, problem arithmetic, arithmetical concepts, attitude and anxiety. In this work the structural approaches of Stern and Cuisenaire are distinguished from those of Dienes as offering a single as opposed to a multiple concrete representation. The approach afforded by the latter proved superior in the dimensions of mechanical arithmetic, arithmetical concepts, attitude and anxiety, especially in the case of dull children, whereas the effects resulting from the use of just a single concrete representation were not very pronounced.¹

Outside of mathematics Allen and Daehling (1968) represented each of three different types of content in three representational modes to sixth grade students. The modes were the figural mode employing realistic drawings, the symbolic mode employing words, numbers and drawings suggestively like a poster, and the semantic mode describing the content in complete sentences. The three content types were chosen because they each seemed most naturally related to one of the modes. They were facts about the sea (figural mode), action sequences such as cooking breakfast (symbolic mode), and the properties of atoms in crystallography (semantic mode). In each mode the content was introduced through a sequence of slides synchronised

1 Biggs (1967), pp 239 - 242. Multiple concrete representation was provided by the Multibase Arithmetic Blocks (MAB) and the Algebraic Experience Materials (AEM) of Dienes. The Multibase Arithmetic Blocks have been described in detail on pp. 50-51 of Chapter 1. The Algebraic Experience Materials are briefly described in Appendix S2 - 4.

with an audio-tape which gave identical verbal introductions with each mode. At the end of the sequence for each content all subjects answered the same multiple choice questions. Despite the bias that had been deliberately introduced the authors of the study concluded that a change in mode did not affect learning significantly.

Even from this small sample of the peripheral literature it appears that the effects of didactic representatives reveal themselves in the case of certain groups of subjects only, namely those which for one reason or another are conceptually immature. Their understanding is strengthened through transition between representations. Although no precise measurement of psychological equivalence emerges from the investigations the importance of providing certain pupils with the opportunity for making such transitions is emphasised, and by implication also the importance of inducing them to recognise those relationships between representations on which their psychological equivalence depends.

Core Literature

Thorndike (1922) presented each of nine algebraic questions ¹ in a familiar and an unfamiliar form. Three of his questions are given in Table 2 - 3.

Three groups were chosen at random from 97 graduate students. All members of the same group worked the same questions, every question being assigned to exactly one group in one of its forms. For each question the percentage

1 In several of Thorndike's questions there was also a slight change in content between the two forms.

Table 2 - 3

A Sample of Thorndike's Questions

Number of question	Familiar form	Unfamiliar form
2	What is the square of a^2x^3 ?	What is the square of $r_1^3 r_{11}^2$?
3	Simplify $4ac + \left(\frac{b^2}{c^2} \times \frac{c^2}{d^2} \times \frac{d^2}{b} \right)$	Simplify $5a^2b + (b^2 \div c^2)(c^2 \div d^2)(d^2 \div b)$
4	There are two numbers. The first number plus 3 times the second number equals 7. The first number plus 5 times the second number equals 11. What are the numbers ?	$y = ax + b$ When $x = 3$, $y = 7$. When $x = 5$, $y = 11$. What does a equal ? What does b equal ?

of wrong or incomplete answers was recorded. On the basis of this scoring system the percentage for the familiar form was markedly lower than for the unfamiliar form in all but two of the questions, where it was almost equal. In question 4 the difference was extreme with 16% in the familiar as against 70% in the unfamiliar form.

It would appear that these subjects had been narrowly trained on specific mathematical tasks in which they could take their cues for certain manipulations from the appearance of the questions without making much of an attempt at their proper analysis. In questions 2 and 3 the changes are superficial changes in notation to which Thorndike's subjects nevertheless could not adjust during the permitted time. The introduction of suffixes ¹ in question 2 caused their performance to decline and the use of division signs within the brackets caused even more deterioration. The unfamiliar form of question 4. requires the fitting of two conditions to an equation in 4 variables before the simultaneous equations obtainable from the familiar form can be reached. Thorndike acknowledges that this change is of a profounder type. The two contexts require very different analyses before the same set of simultaneous equations is reached.

Wilkins (1928) used 80 undergraduates ² to study changes in performance of syllogistic reasoning when

1 Chapter 1, p. 45. The suffixes and indices are both numbers whose differing functions the learner has to distinguish.

2 An earlier study of hers had suggested to Wilkins that greater reliability would be obtained from a highly select group of subjects.

syllogisms were presented to each subject in four different representations. A sample question illustrating the four representations is given in Table 2 - 4. Representation A expresses the syllogism in words drawn from everyday language, representation B uses symbols, and representation C uses words which are either meaningless ¹ outside this particular context or are drawn from some highly specialised scientific vocabulary. Representation D again uses familiar words, but this time many of the conclusions are statements which the subjects would know to be factually correct, but which do not necessarily follow from the premises. Wilkins calls the representation 'suggestive'. Factual truth is to suggest inferential validity and subjects are assessed on their ability to distinguish between the two.

Subjects were divided into four equal groups and took the four parts A, B, C and D of the test, one part for each representation, in different orders. Each part ² consisted of 20 syllogisms with three conclusions, to be marked true or false. The order of the questions with the same content varied from part to part.

Through variation of the order in which subjects took the four parts of the test Wilkins sought to control the practice effect. She had found an increase of 5 points in the mean score of subjects who took part A or part D

-
- 1 Such words functions like the symbols x, y, z, which only become invested with meaning by the context.
 - 2 There was also a parallel test of the same length in each representation, so that reliability could be assessed.

Table 2 - 4

A Sample of Wilkins's Questions

A The school is west of the post-office; the court-house is west of the school; therefore a. the court-house is west of the post-office. b. the court-house is east of the post-office. c. the post-office is east of the court-house.	C Crogingalong is west of Boggabilla; Pitarpunga is west of Crogingalong; therefore a. Pitarpunga is west of Boggabilla. b. Pitarpunga is east of Boggabilla. c. Boggabilla is east of Pitarpunga.
B X is west of Y, Z is west of X; therefore a. Z is west of Y. b. Z is east of Y. c. Y is east of Z.	D Chicago is west of San Francisco; Reno is west of Chicago; therefore a. Reno is west of San Francisco. b. Reno is east of San Francisco. c. San Francisco is east of Reno.

last as against those who took it first. Parts B and C were never placed first or last and the gain arising from the practice effect was only 3 points in their case.

The correlations ranged from 0.69 between parts A and D to 0.81 between parts B and C. From this Wilkins concluded that there was a high degree of relationship between the four parts.¹ The correlations were, however, lower than the correlations between the tests and their parallel forms in each representation. Part A had by far the greatest number of scores between 90 and 100 per cent. This is explained by Wilkins as due to the fact that the other three parts "all involve more abstraction, more ability to resist distraction than part A does. They are all less familiar than part A and involve less habitual reactions".

Besides correlations Wilkins provides frequency distributions and mean scores for each representation. It is in the latter two rather than in the correlations that a measure of psychological equivalence should be sought.²

The elements by which the four representatives are differentiated are exchangeable on a one : one basis. Given this type of difference there would have been little value in asking for a translation between two representations. In fact, the simultaneous presence of such representations

1 No tests of significance were made.

2 The mean scores out of 80 for the four representations A, B, C and D were 50.5, 44.8, 41.3 and 46.6 respectively. No tests of significance were made.

would have allowed subjects to pass between the representations by making simple substitutions¹ on a one : one basis and then to think their way through each syllogism in the representation they preferred.

Outside of mathematics Ramsay (1965) investigated the attainment of concepts from figural and verbal representations. In the figural representation 6 two-valued attributes of figures and their borders appeared in different combinations on cards, and in the verbal representation these same combinations were given in words. Concepts were defined by a combination of two attributes such as green and circular, so that there would be 16 out of a total of 64 cards exemplifying each concept. The experimenter selected one of these 16 cards and asked subjects to find the concept by trying other cards in sequence and being told at each trial whether the card selected did or did not contain the required combination. Among the large number of results obtained was the information that subjects took more time to attain the concepts with the verbal than the figural material, also that individuals working in pairs needed more trials with the figural material whereas for single subjects this situation was reversed. The important features of Ramsay's study are the tight control of the content and the choice of a design which makes possible the detailed study of the learner's progress in each representation.

Fennema (1969) used 95 children aged 7 - 8 years to compare the relative effectiveness of a symbolic and a

1 cf. Chapter 1, p.54.

concrete representation in the teaching of a mathematical principle. Contending that discrepancies in studies designed to assess the value of sensory aids might possibly be explained through differences in the amount of meaningful learning that occurred rather than through differences in representation she designed her thesis to prove that the meaningful learning of a mathematical principle could result from a symbolic and a concrete representation equally well. Her six hypotheses are listed in Table 2 - 5.

All children admitted to the experiment had to pass a qualifying examination on cardinal number, equivalent sets, numeral recognition and the addition of one digit numbers. After that they were randomly assigned to three treatments for an instructional period extending over 14 days. The objective of all three treatments was to teach them single digit multiplication of products between 1 and 10.¹ Questions were asked in the three forms

- (a) 3,2 $\longrightarrow$ _____
(b) 3,___ $\longrightarrow$ 6
(c) __,2 $\longrightarrow$ 6

The children in Treatment III received no instruction in the principle of multiplication but learnt individual multiplications mechanically. The children in Treatment I were given this principle in symbolic form. They were

¹ The product 1 x 1 was omitted.

Table 2 - 5

Fennema's six Hypotheses

Hypothesis I: There are no significant differences in the learning of a mathematical principle between groups of children who have learned that principle using a meaningful symbolic or a meaningful concrete model.

(The concrete model was the Cuisenaire apparatus.)

Hypothesis II: There are no significant differences in the recall of instances of a selected mathematical principle between groups of children who have learned that principle using a meaningful concrete or a meaningful symbolic model.

(The concrete model was the Cuisenaire apparatus.)

Hypothesis III: There are no significant differences in the ability to solve problems of untaught instances of a mathematical principle between groups of children who have learned that principle using a meaningful symbolic or a meaningful concrete model when they use as aids in problem solving that model with which they learned.

(The concrete model was the Cuisenaire apparatus.)

Table 2 - 5 continued

Hypothesis IV: There are no significant differences in the ability to solve problems of untaught instances of a mathematical principle between groups of children who have learned that principle using a meaningful symbolic or a meaningful concrete model when they have familiar concrete materials to use as aids in problem solving.

(The concrete model was a set of counters.)

Hypothesis V: There are no significant differences in the ability to demonstrate a mathematical principle on an unfamiliar concrete model between groups of children who have learned the principle using a meaningful symbolic or a meaningful concrete model.

(The concrete model was a balance beam.)

Hypothesis VI: There are no significant differences in the ability to demonstrate a mathematical principle on an unfamiliar concrete model between groups of children who have had instruction in the principle with a meaningful concrete or a meaningful symbolic model and groups of children who have received no instruction in that principle.

(The concrete model was a balance beam.)

taught

3,2 $\longrightarrow$ _____

as meaning

2 + 2 + 2 = _____

The children in Treatment II were given the same principle in concrete form with the Cuisenaire rods.¹ After they had familiarised themselves with the number values of these rods by building staircases they were told that finding the answer to

3,2 $\longrightarrow$ _____

meant finding the rod which was the same length as 3 rods of length 2 units placed end to end.

Treatment III thus differed from the other two treatments in the amount of meaning with which the instruction was invested. Treatments I and II differed from each other only in the way in which the principle providing this meaning was presented.

The children took forms of the Recall Test² during and after their respective treatments. After the Recall Test the children took three Transfer Tests. In Transfer Tests I and II the products in the multiplications consisted of numbers from 11 to 16 instead of from 2 to 10. In Transfer Test I the children in Treatment I were again given the multiplication principle in symbolic form and the children in Treatment II were allowed to use the

1 Appendix S2 - 2.

2 Since each product could be presented in one of three forms a sufficient variety of examples was to hand.

Cuisenaire rods. In Transfer Test II the children in Treatments I and II again answered questions of the same type, but the assistance given to them this time consisted of counters which they could use at discretion. The children in Treatment III did not take Transfer Tests I and II.

In Transfer Test III the children in all three treatments were required to demonstrate their knowledge of the products from 2 to 10 by hanging weights on a balance beam. The test was conducted on an individual basis.¹ The tester would either place several weights of the same size on his side of the beam and ask the child to place the single equivalent weight on the other side; or he would use a single weight on his side, hand the child several weights of the same size and ask him to balance the beam.

For Hypothesis I the children in Treatments I and II were compared for their performance on questions anticipating questions in the Recall Test. For Hypothesis II the same children were compared for their performance on the actual Recall Test. There were no significant differences² and both Hypotheses were accepted.

For Hypotheses III and IV the scores obtained by children in Treatments I and II were compared on Transfer Tests I and II respectively. Both Hypotheses were rejected

1 Testing in a concrete representation requires either a very small number of subjects or a very large number of testers, since the results of any one performance in a succession of performances can not be held fast for later scoring without the use of very expensive equipment.

2 .05 was the level selected for the rejection of all the hypotheses.

in favour of the children who had not used Cuisenaire, the result for Hypotheses IV being highly significant.¹

For Hypothesis V the scores on Transfer Test III of the same children were again compared, but this time no significant difference could be detected.

For Hypothesis VI the combined scores on Transfer Test III of the children in Treatments I and II were compared with the scores of the children in Treatment III, and again no significant difference was observed.

This study resembles the conventional type of comparison in that the criterion performance required use of the symbolic ² representation from all subjects except in Transfer Test III, where the calculations had to be set up on the balance beam. With the content of the calculations held constant the latter test could thus have provided a measure of psychological equivalence if results had been compared with the results of the Recall Test. Fennema's objective, however, was to show that the understanding of

1 For Hypothesis III the probability of obtaining the observed differences in the results by chance was actually .053, but for Hypothesis IV it was only .003.

2 The Cuisenaire rods and counters only served as additional aids in some of these performances. The markedly lower scores (Hypothesis IV) obtained by the Cuisenaire children in Transfer Test II seem difficult to explain, especially as the use of the counters was not obligatory in this test. It seems possible that the Cuisenaire children felt that they had to master the counters as a second concrete representation and fell behind the other group as a result of this pre-occupation, or that the two concrete representations somehow interfered with one another.

the principle of multiplication was more important than the use of a concrete representation, and to ascribe success¹ with the Cuisenaire rods in other experiments to the greater amount of meaning which teachers using these rods impart to their lessons rather than to the perceptual effects of the rods as such. Her results confirm that teaching with symbols alone can be made as meaningful as teaching with the Cuisenaire. But there are no results to show differing effects resulting from meaningful and non-meaningful teaching as defined in the experiment, since the children in Treatment III did not fall behind the others in Transfer Test III and never participated in the other Transfer Tests.

Farris (1970) selected 24 fifth grade students to assess their transfer abilities between the enactive, iconic and symbolic modes of representation. The subject matter was equivalence of fractions.² The enactive representation was provided by the Cuisenaire rods, the iconic representation by drawings of rectangular strips containing subdivisions and shadings, and the symbolic representation by pairs of integers functioning as numerator and denominator in the conventional way.

The three representations were denoted by the capitals E, I and S. A group of questions forming an objective was

-
- 1 Cf. Table 2 - 1 supra. Fennema's children were in the age group where use of the Cuisenaire rods tends to improve performance.
 - 2 The topic of equivalent fractions naturally lends itself to transfer tasks within one and the same stimulus material. In the iconic representation, for example, 3 strips shaded out of 9 represent the same rational number as 1 strip shaded out of 3.

then labelled by a pair of these capitals, with the first capital denoting the representation of the question or input, and the second the representation of the answer or output. A sample question from an (I,E) objective is given in Table 2 - 6. Objectives were divided into antecedent and consequent according as they had to be mastered with or without instruction. The mastery¹ of a pair of antecedent objectives was a pre-condition for being tested on a consequent objective. Four of the hypotheses constructed around these objectives are listed in Table 2 - 7.

In this design there are two types of transfer, one within each objective and the other between objectives. In the first type the change in representation is accompanied by a switch from question to answer. There is thus no way of telling whether failure to perform arises from the nature of the question or from a change in representation. In the second type the change in performance between, say, an (I,I) and an (I,S) objective is due to the representation alone, provided the questions in the two objectives are identical or at least of comparable difficulty.²

Hypothesis No. 3 was the only one among all those formulated which the experiments confirmed. This would seem to indicate the presence of psychological inequivalence between the representations despite the fact that the children were relatively mature in age and had demonstrated mastery of all the representations involved.

1 Mastery was defined as 80 per cent of the questions correctly answered.

2 This balance in content was provided.

Table 2 - 6

A Sample Question from Farris's experiments asking for an enactive output from an iconic input.

(The symbolic representation of the fractions is also present, but only to define the meaning of the Cuisenaire rods.)

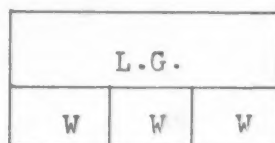
Suppose the blue (B) rod stands for 1
then the light green (L.G.) rod stands for $\frac{1}{3}$
and the white (W) rod stands for $\frac{1}{9}$

Use your rods to show that the two fractions suggested by the picture are equivalent.



Tell me when you have your answer ready.

(When he signals that he is ready, check his arrangement. It should look like this)



(Did the student respond correctly?)

no yes

Table 2 - 7

Four of the Hypotheses advanced by Farris

No. 2.	If a student demonstrates mastery of (I,S) and (S,I) objectives from the same cluster where at least one of these objectives was acquired as a result of explicit instruction, he will master an (S,S) objective from the same cluster without explicit instruction.
No. 3.	The order in which a subject acquires (I,S) and (S,I) objectives from a particular cluster will not significantly affect the implicit mastery of an (I,I) or (S,S) objective from the same cluster.
No. 5.	If a subject demonstrates mastery of (E,S) and (I,S) objectives from the same cluster where at least one of these objectives was acquired as a result of explicit instruction, he will master an (E,I) objective from the same cluster without explicit instruction.
No. 6.	If a subject demonstrates mastery of (E,S) and (I,S) objectives from the same cluster where at least one of these objectives was acquired as a result of explicit instruction, he will master an (I,E) objective from the same cluster without explicit instruction.

The main contribution which this study makes to the measurement of psychological equivalence is that subjects are confronted by two representations at one and the same time. Explicit translation as a component in this measurement is suggested by the design but imperfectly developed in that the switch from one representation to the other is also accompanied by a switch from question to answer in each case.

Summary

This chapter has surveyed the widespread interest that exists in the comparison of representations together with the diversity of the experimental designs on which these comparisons have been based. Those designs whose contributions to the study of psychological equivalence seemed distinctive were analysed in greater detail. It appears that they are much smaller in both number and extent than the conventional ones encountered in the main stream of research, and that the measurement of psychological equivalence they have to offer is only a partial one when judged against the definitions of this measurement as developed in Chapter 1.

CHAPTER 3

THE DEVELOPMENT OF REPRESENTATIONS OF THE TETRAHEDRAL GROUP

Introduction

To develop tests suitable for use in small experiments that would extend over no more than two classroom periods a mathematical structure was selected which

- (1) was unfamiliar to subjects,
- (2) possessed operators,
- (3) was small enough to be adequately sampled,
- (4) was large enough to avoid duplication of the same questions in teaching and testing,
- (5) could easily be represented in a large number of perceptual forms,
- (6) admitted of changes in representation resulting from a change in the operators.

All these requirements are met by the structure known as the tetrahedral group. Since this structure is not as a rule studied at the school level no biases from any specific classroom approach would be present in subjects. Being a group the structure has an operation, and this also serves to test knowledge operationally in the psychological sense. The number of its elements is only 12, but since the group operation makes it possible to reach any element from any other there are 144 possible

tasks¹ for subjects to perform. Because of its smallness the structure does not require a representation with an elaborate rule system, so that a variety of didactic representations is easily created. And since the transition from one element to any other can be effected in more than one way an alternative² dimension for changing the representation is to hand.

The mathematics of the tetrahedral group

Figure 3-1 shows the model of a regular tetrahedron - a solid having four congruent equilateral triangles as its faces. Suppose a box were constructed into which this tetrahedron fitted exactly. Because of its symmetry the tetrahedron could be placed into this box in 12 different ways. The number 12 is arrived at as follows. If any one vertex were chosen and assigned a position in the box then any one of the remaining three vertices in the equilateral triangle opposite the chosen vertex could take up 3 different positions in this box. For the remaining two vertices there would then be no further choice. Since the first vertex can be chosen in 4 ways there are altogether $4 \times 3 = 12$ different positions of the solid arising from 4 possible choices for the first vertex and 3 possible choices for the second vertex.

¹ The number 144 includes the identity tasks which start and end at the same element.

² This alternative dimension has already been discussed in Chapter 1, p. 52.

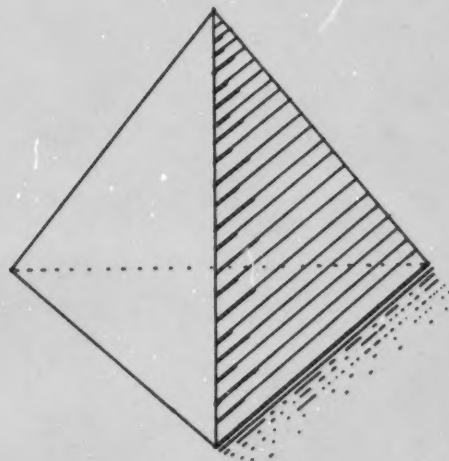


Figure 3 - 1 The Regular Tetrahedron with one of its faces shaded.

If the tetrahedron is changed from any one position inside the box to any other position it is subjected to a transformation in space. This transformation is special ¹ in that every point in space ² that was occupied by a point of the tetrahedron in the first position is again occupied by a point of the tetrahedron in the second position. In general the two points occupying the same point in space will not be the same points of the tetrahedron. If one vertex were returned to its original position, then the point of this vertex would be an exception. If two vertices were returned to their original position then

1 Many other transformations are possible. / For example, the tetrahedron could be moved to another box.

2 It is assumed that the box remains fixed in space.

there would be no further choice for the remaining two vertices. Consequently all the points of the tetrahedron would return to their original positions and the transformation would be called the identity transformation.¹

Since there are 12 possible positions there are also 12 possible transformations inclusive of the identity transformation. This set of 12 transformations is known as the symmetry group of the tetrahedron because it possesses the four properties by which a group structure is defined:

- (1) Any two transformations performed in succession are equivalent to one of the 12 possible transformations. This is the property of closure.
- (2) If three such transformations are performed in succession then one can replace either the first two or the last two by the single transformation to which they are equivalent without altering the overall effect of the transformations. This is the associative property.
- (3) In the set of 12 transformations there is an identity transformation which returns the tetrahedron to its original position. This is the property of the identity.
- (4) Whenever a transformation has been made there is always a single transformation which will return the tetrahedron to its original position. This is the

¹ Because the initial and final positions are identical. In the addition of numbers the addition of 0 is the identity transformation since it leaves every number unchanged.

property of the inverse.

Many groups possess an additional fifth property, the commutative property. This property states that the order in which transformations are performed is immaterial. For example, if the number 7 is transformed by the addition of 2 followed by the addition of 3 to become 12, then it would also become 12 if it were transformed by the addition of 3 followed by the addition of 2. The transformations of the tetrahedral group do not possess this property, and its absence could be expected to add to the unfamiliarity of the tasks for subjects in the experiment.

Up to this point the property of closure has been established from perfectly general considerations of the transformations. By themselves these considerations do not, however, allow us to determine to which single transformation any two given transformations are equivalent. It is as though we knew that any number added to any other number will give a number for an answer, but had not constructed an addition table showing the number combinations in detail. In order to construct such a table for the transformations of the tetrahedron we must describe these transformations in detail. One way¹ of doing this is to label the four positions which the vertices of the tetrahedron can occupy in the box as A, B, C, D, and then to describe a given transformation as

¹ An alternative way would be to label the vertices of the tetrahedron and then to describe a given transformation as taking vertex A to where vertex B was, vertex B to where vertex C was, and so on. In the first case the frame of reference lies in space, in the second case it lies in the moving solid.

taking the vertex in position A to position B, the vertex in position B to position C, and so on. Every transformation is then a permutation¹ of the letters A, B, C, D. If the 12 transformations are also given identification labels based on the permutations then they can all be combined in pairs to give a single transformation, and all the combinations can be recorded in a table of $12 \times 12 = 144$ entries.² This table, which is commonly referred to as the 'multiplication' table of the tetrahedral group, is given in Table 3 - 1. The transformations have been denoted by T_1 to T_{12} .

In the above table the linking of the transformations in pairs is a binary³ operation performed on the transformations. But to know the result of any one such linking we have to perform two permutations of the symbols A, B, C, D in succession and note their combined effect. Any such permutation is a unary⁴ operation performed on

-
- 1 The tetrahedral group is in fact the group of all even permutations. See Ledermann (1964), p.91.
 - 2 Other group tables of size 12×12 also exist. The addition table for the clock arithmetic of twelve, for example, has an entirely different structure.
 - 3 A binary operation links together two elements so as to form a third element of the same kind. Addition may be thought of as a binary operation linking two numbers so as to form a third number.
 - 4 A unary operation is performed on a single element and changes this element into another element. Here the element consists of the collection of letters A, B, C, D which are permuted. Number addition can be thought of as a unary operation if the first number is regarded as the element and the second number together with the operation of addition as the unary operation. Instead of thinking of addition as "ADD the numbers 2 and 3" we now think of it as "ADD 3 MORE to the number 2". For further details see Dienes (1971), who regards the unary interpretation as being psychologically simpler.

Table 3 - 1

The Multiplication Table for the Transformations of
the Tetrahedral Group

	T ₁	T ₂	T ₃	T ₄	T ₅	T ₆	T ₇	T ₈	T ₉	T ₁₀	T ₁₁	T ₁₂
T ₁	T ₁	T ₂	T ₃	T ₄	T ₅	T ₆	T ₇	T ₈	T ₉	T ₁₀	T ₁₁	T ₁₂
T ₂	T ₂	T ₄	T ₈	T ₁	T ₁₀	T ₁₁	T ₆	T ₁₂	T ₅	T ₉	T ₇	T ₃
T ₃	T ₃	T ₅	T ₁	T ₇	T ₂	T ₉	T ₄	T ₁₁	T ₆	T ₁₂	T ₈	T ₁₀
T ₄	T ₄	T ₁	T ₁₂	T ₂	T ₉	T ₇	T ₁₁	T ₃	T ₁₀	T ₅	T ₆	T ₈
T ₅	T ₅	T ₇	T ₁₁	T ₃	T ₁₂	T ₈	T ₉	T ₁₀	T ₂	T ₆	T ₄	T ₁
T ₆	T ₆	T ₈	T ₉	T ₁₀	T ₁₁	T ₁	T ₁₂	T ₂	T ₃	T ₄	T ₅	T ₇
T ₇	T ₇	T ₃	T ₁₀	T ₅	T ₆	T ₄	T ₈	T ₁	T ₁₂	T ₂	T ₉	T ₁₁
T ₈	T ₈	T ₁₀	T ₂	T ₆	T ₄	T ₅	T ₁	T ₇	T ₁₁	T ₃	T ₁₂	T ₉
T ₉	T ₉	T ₁₁	T ₆	T ₁₂	T ₈	T ₃	T ₁₀	T ₅	T ₁	T ₇	T ₂	T ₄
T ₁₀	T ₁₀	T ₆	T ₇	T ₈	T ₃	T ₁₂	T ₅	T ₉	T ₄	T ₁₁	T ₁	T ₂
T ₁₁	T ₁₁	T ₁₂	T ₅	T ₉	T ₇	T ₂	T ₃	T ₄	T ₈	T ₁	T ₁₀	T ₆
T ₁₂	T ₁₂	T ₉	T ₄	T ₁₁	T ₁	T ₁₀	T ₂	T ₆	T ₇	T ₈	T ₃	T ₅

A procedure for constructing this Table is described in Appendix S3 - 1 on p. 218. See also Appendix S3 - 5 on p. 235.

the symbols and therefore logically¹ prior to the binary operation.

Because of the close connection that exists between the two types of operation it is possible to re-interpret the multiplication table in terms of unary operations. In that event we have to regard the 12 symbols in the outer column on the left as positions of the tetrahedron, the 12 symbols in the top row outside the table as unary operators, and the 144 entries inside the table again as positions reached after the unary operations have been performed. Since T_1 is the identity transformation² in the binary interpretation of the table it is natural to regard T_1 as the starting state or neutral position in the unary interpretation. The positions denoted by the 11 remaining suffixed letters are then the positions reached from the starting state by the application of the transformation whose letter they bear.

Finally, if each position of the tetrahedron is regarded as a state of the system it is possible to give the table a third interpretation, in which two given states are joined by a binary operation so as to form a

¹ This priority is readily seen in the construction of the cardinal numbers. Given the number 0 we ADD 1 MORE to this number so as to obtain the number 1, and then reach numbers of arbitrary size by the continued application of the unary operation.

² An inspection of the entries in the table shows that the transformation T_1 leaves unchanged every transformation with which it is linked.

third state. In the description of the symmetries of the tetrahedron such an interpretation does not, however, seem a very natural one.¹

Frequently in mathematics one looks for economical ways of describing what is going on. In referring to points in the plane, for instance, it is common practice to choose two directions at right angles to each other and then refer to each point in terms of these two directions by means of ordered number pairs. There are many possible choices for the privileged directions, but

¹ The following three examples, based on Dienes (1971), will help to make clear the three interpretations for the cardinal numbers:

- (i) Binary operation linking two numbers regarded as operators.

6 MORE followed by 3 MORE = 9 MORE

This result will hold regardless of the size of the number to which it is applied.

- (ii) Unary operation performed on a single number regarded as a state.

6 subjected to an increase of 3 MORE = 9

- (iii) Binary operation linking two numbers regarded as states.

6 and 3 ADDED together = 9

There is one other important difference between the numbers and the states (positions) of the tetrahedron. In the case of the tetrahedron the 12 states are known before the group table for the transformations is constructed. In the case of the cardinal numbers, on the other hand, the numbers are known only after the transformation ADD 1 MORE has been defined.

spatial properties such as the distance between two points or the area of a given triangle are not affected by any particular choice of these directions.

In the case of the tetrahedron and its symmetry group we have an analogous situation. From among the 11 possible transformations¹ it is possible to choose just a few and then to describe all the remaining transformations in terms of them. If the choice is to be as economical as possible then the chosen transformations must be sufficient in number to describe all the others and also not be describable in terms of each other.²

The choices which offer themselves fall into two classes consisting of two transformations each. In one class both transformations are of the same type. They are each rotations through 60° about an axis³ running through one of the vertices and through the centroid⁴ of the equilateral triangle forming the opposite face. Two possible axes of this kind are shown in Figure 3-2. Since there are three vertices in every face opposite to a given

¹ The identity transformation is not a possible choice, since no other transformation can be described in terms of the identity, and since the identity yields no new transformation when it is combined with any other transformation.

² Such transformations are said to form an irredundant set of generators for the group.

³ These rotations may be clockwise or anticlockwise, when viewed in a direction running from the vertex to the opposite face.

⁴ The centroid is the meeting point of the three lines joining each vertex of the equilateral triangle to the midpoint of the opposite side. In the case of an equilateral triangle the centroid is also the point of rotational symmetry.

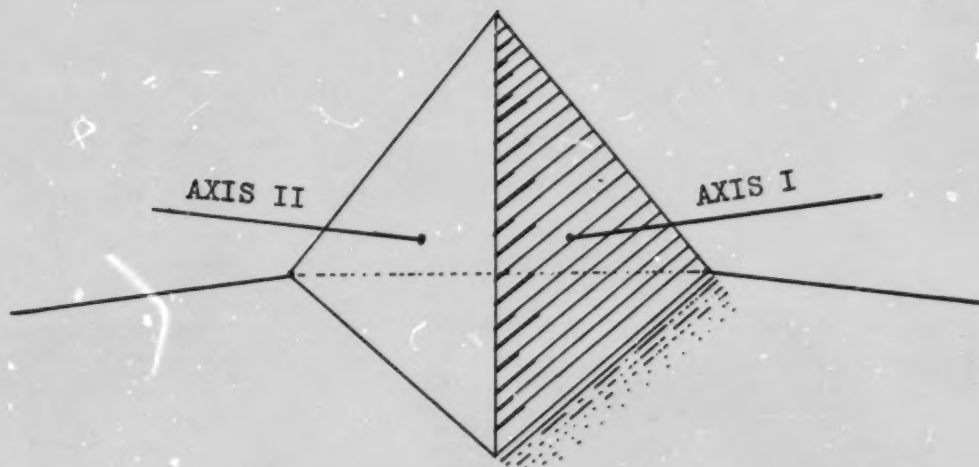


Figure 3 - 2. All positions of the tetrahedron can be reached by suitable combinations of rotations about Axis I and Axis II.

vertex each of the two rotations will bring the tetrahedron back to its original position if it is performed three times in succession. Both transformations are therefore of order three and this class of transformations will be referred to as being of type $(3,3)$.

In the second class one of the transformations is again a rotation of the kind just described. The other transformation is a rotation about an axis joining the midpoints of two edges of the tetrahedron which do not meet at a vertex. Two possible axes of this kind are shown in Figure 3 - 3. The second rotation interchanges all the vertices in pairs and is therefore of order two. This class of transformations will be referred to as being of type $(3,2)$.

It may be verified from the group table ¹ that all transformations other than the identity transformation ² are either of order three or of order two, and that any pair of transformations of order two commute. Because of this last mentioned property it is not possible to express a transformation of order three as a combination of transformations that are all of order two. A type (2,2) representation is thus impossible and no types other than those mentioned exist.

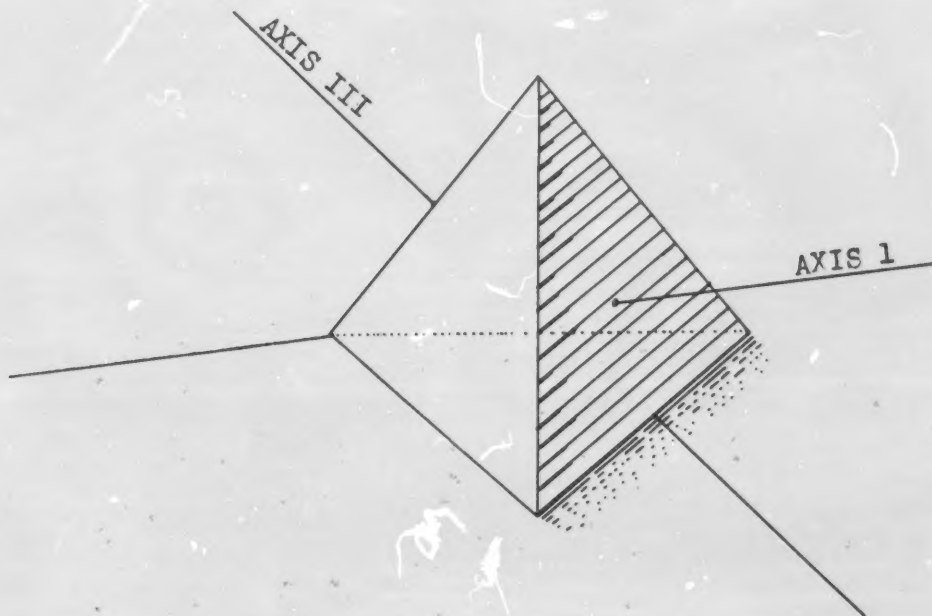


Figure 3 - 3. All positions of the tetrahedron can be reached by suitable combinations of rotations about axis I and axis III.

1 p. 95 supra.

2 The identity transformation is of order one in all groups.

The development of the representations

For each of the two types of transformation representations were developed by means of unary operators known as Machines. For each type the stimulus material¹ consisted of algebraic symbols, cardinal numbers, geometric shapes in frames, and meaningful sentences. There were thus eight representations known as Games. The transformation type, stimulus material and code letter for each Game are listed in Table 3 - 2.

Table 3 - 2

The Transformation Type, Stimulus Material and Code Letter
each Game

Class of Transformation	Stimulus Material	Game
Type (3,2)	Algebraic Symbols	A
Type (3,2)	Cardinal Numbers	B
Type (3,2)	Geometric Shapes in Frames	C
Type (3,2)	Meaningful Sentences	D
Type (3,3)	Algebraic Symbols	E
Type (3,3)	Cardinal Numbers	F
Type (3,3)	Geometric Shapes in Frames	G
Type (3,3)	Meaningful Sentences	H

1 Considerations of cost and administration unfortunately precluded the use of colour and of a three-dimensional model of the tetrahedron as stimulus material. A model was, however, shown to subjects in the introductory talk and related to the Games they were to learn.

The two representations using algebraic symbols were conventional, except that in the unary interpretation the same symbols served both as states and as operators. The two iconic representations using shapes in frames are very close to any 2 or 3 dimensional representation of the tetrahedron itself, with the vertices shown in different positions. The remaining four representations used either familiar numbers and number transformations, or familiar sentences with familiar grammatical transformations. In both cases the rules governing the familiar transformations had to be supplemented by special rules stating which transformations were to be applied to which number or sentence. Without these conditional rules¹ the requirements of the group structure could not have been fulfilled, but with them the structures of cardinal number and of grammar could be enlisted to serve as substructures in the representations.

The stimulus material and function of the two Machines effecting the Transformations are fully described and illustrated for each Game in the Special Appendices to the present Chapter.² In each case the function of the Machines is described in words, and for each Machine there is provided a chart known as Machine Reminder, on which all states and transformations are displayed.

1 Conventional mathematical representations are generally free from such conditional rules, but in ordinary language such rules are frequently found to control the meanings of words in their context.

2 pp. 217 - 249.

For the experiments reported here an Initial Learning Test was developed for Game B, and Transfer Tests between Game B and all the other Games except Games E and H.¹ These tests and their development are fully described in Chapter 4.

¹ These two unused Games were included here for the sake of completeness.

CHAPTER 4

THE DEVELOPMENT OF THE TESTS FROM THE REPRESENTATIONS.

The psychological equivalence of pairs of representations of the tetrahedral group was assessed by means of specially designed transfer tests. But in order to be capable of reacting to these transfer tests positively subjects had to undergo a preliminary period of instruction. During this period they were familiarised with the overall objective of the investigation, and also with the structure of the tetrahedral group in one of its representations. This representation was taught in considerable detail with an Initial Teaching Booklet written in the form of a linear programme.¹ The tester worked through the booklet with the class and then immediately asked them to write the Initial Learning Test, which served as a qualifying examination.² This completed the first class period of 50 minutes. The Transfer Test was given one week later during a second period of the same length. The introduction to this test was only the minimum required for understanding the description of the two Machines in the new representation and for matching up the Machine Reminder charts of the two representations correctly.

1 This booklet is an adaptation from a branch programme written by the present author in 1973 at the University of Sherbrooke, Quebec, from specifications prepared by Dr. S.Klein of the Technical University of Budapest. The branch programme was intended to lead children to the acquisition of information about mathematical structures in a limited time, and the test given immediately after its reading was intended as a dynamic approach to the measurement of intelligence as proposed by Vygotsky (1962, p.103).

2 All pupils satisfied the criterion of 60%.

For every experiment it was thus necessary to have to hand two types of test, the Initial Learning Test and the Transfer Test. Besides these tests there was also required a Model of the Tetrahedron for demonstration, an Initial Teaching Booklet, and a Sheet of Opening Instructions through which this booklet was introduced. Each of these five items will now be discussed in turn.

The Test Materials

1. Model of the Tetrahedron

This was intended for use with the Opening Instructions, to provide a brief introduction to the mathematical background of the Games. It was constructed¹ from four white plastic rods meeting three at a time in four perforated plastic spheres. The spheres were coloured red, blue, green and yellow. Apart from the coloured spheres, which merely served as labels for the vertices, the appearance of the model is suggested by Figure 3 - 1 in the previous chapter.²

2. Opening Instructions

These were designed to engender in subjects the right attitude before their participation, to provide as much background as possible in the allocated time, and to ensure uniformity in administration between one group and the next.

1 The model was assembled from a Hungarian version of the Kugeli kit known as Babylon. The pieces in this kit are larger than those of the Kugeli kit (E.S.A. 1976) but resemble them in most other respects.

2 Page 91.

They are reproduced in their entirety in Special Appendix ¹
S 4 - 1.

3. The Initial Teaching Booklet for Game B

This booklet ² was designed to include a representative sample of all the different types of task that can be encountered in a type (3,2) representation of the tetrahedral group. An inspection of either of the Machine Reminder charts ³ for Game A shows that there are 11 proper tasks requiring 1, 2, 3, or 4 steps for their completion. To these must be added the improper tasks X^3 and Y^2 . Since their completion leads back to the starting position they are equivalent to the identity transformation, which may also be conceived as a task of 0 steps.⁴ There are thus 12 tasks in all, as listed in Table 4 - 1.

The number of steps served as one criterion for differentiating between tasks. A second criterion was provided by the number of decisions a subject had to make in order to complete his task. If he was presented with a starting position and a machine sequence for reaching the final position he just had to apply the machines correctly without making any decisions. On the other hand, it was

1 Page 251.

2 A copy of this booklet and of all the tests will be found in the General Appendices.

3 Appendix S3 - 1.

4 Strictly speaking the identity transformation should be conceived as a task of 0 steps, in keeping with the practice of expressing all other tasks in their shortest form. Improper tasks requiring the use of machines were however introduced into the tests on two occasions, in question 6 of the Initial Learning Test and in question 6 of every Transfer Test between a Game of Type (3,2) and a Game of Type (3,3).

Table 4 - 1

The different Tasks available in a Game of Type (3,2),
described by the symbols of Game A.

Task	No. of Steps
x^3 or y^2	3 or 0, or 2 or 0
X	1
Y	1
x^2	2
XY	2
YX	2
yx^2	3
XYX	3
YXY	3
x^2y	3
xyx^2	4
x^2yx	4

possible to present him with the starting position and the final position, and to ask him to reach this final position in the smallest number of steps. The latter alternative then offered him a task with decision points equal in number to the number of steps.¹ On the assumption that task

1 In the case of an improper task the number of steps is variable. Such tasks occurred in question 6 of the Initial Learning Test, in question 6 of the Transfer Test between Game B and Game F, and in question 6 of the Transfer Test between Game B and Game G. Each time the number of steps intended to produce the answer was indicated by the number of lines that were left blank. See the General Appendices.

difficulty increased with the number of decision points, all tasks requiring more than 2 steps were always presented in the form which required the decisions to be made, since in the other form they would have been simply an aggregate of 1 step tasks.

In accordance with these principles a representative selection of tasks comparable in difficulty¹ could be obtained. Table 4 - 2 contains a complete analysis of the tasks used for the demonstrations and exercises in the Initial Teaching Booklet, which was developed from the representation of Game B.

-
- 1 Alternate questions in every Transfer Test always commenced with tasks in different representations. To ensure that the two halves of the test created in this way were compatible in difficulty when the effect of this difference was discounted, a special version of the Transfer Test between Game B and Game C was administered to 50 pupils. In this version the first task of every question was given in the representation of Game B, and the translated or second task in the representation of Game C. Wilcoxon tests (Siegel 1956, pp 75 - 83) comparing performance in the odd numbered questions with performance in the even numbered questions yielded p values of .99 and .90 for Games B and C respectively. See p. 112 infra for a detailed description of the Transfer Tests.

The Task Analysis of the Initial Teaching Booklet for Game B

(In each task the part which subjects were asked to complete has been placed between asterisks.)

<u>1. Demonstrations</u>		
<u>Starting position</u>	<u>Machine operation(s)</u>	<u>Final position</u>
20	TWO-FOUR	5
6	TWO-FOUR	12
3	SWITCH	1
5	SWITCH	15
4	SWITCH	60
20	SWITCH	12
<u>2. Exercises</u>		
<u>Starting position</u>	<u>Machine operation(s)</u>	<u>Final position</u>
20	TWO-FOUR	* 5 *
6	TWO-FOUR	* 12 *
10	SWITCH	* 2 *
20	SWITCH	* 12 *
60	* TWO-FOUR *	15
12	* SWITCH *	20
15	TWO-FOUR/SWITCH	* 6 *
15	* SWITCH/TWO-FOUR*	10
3	* TWO-FOUR/ SWITCH/TWO-FOUR*	60
1	* TWO-FOUR/SWITCH TWO-FOUR/TWO-FOUR*	5

4. The Initial Learning Test for Game B.

The Initial Learning Test consists of a series of 10 questions resembling but not identical with the questions of the Initial Teaching Booklet. The analysis of the tasks occurring in these questions is given in Special Appendix S4 - 2. The test was written as soon as the reading of the Initial Teaching Booklet had been completed. While writing their answers subjects were allowed to retain the Booklet, the Machine Description and the two Machine Reminder charts for reference.

The scoring system of this test provides for the award of 20 points in all, 1 point for each number or machine name placed correctly on one of the empty lines. In a task involving more than 1 step no points were awarded for correct answers following upon an incorrect answer in one and the same sequence of steps. In question 8 of Table 4 - 3, for example, a subject who put an incorrect answer on the first empty line received a score of 0 for the question, regardless of what he put on the other two lines. Again, in question 8 the answer actually consists of 2 steps, but provision is made for the award of 1 extra point for leaving the 3rd line empty. Question 9 is introduced in the same way as question 8 but requires 3 steps for completion. These two questions together with question 10 were thus allocated 10 out of the total possible of 20 points. 7 points were allocated to the last two questions, which were assumed to be the most difficult.

Table 4 - 3

Two Sample Questions from the Initial Learning Test

8. Your starting number is _____ → 1

If you wish to change this number into _____ → 6

you will use this machine _____ →

and then this machine _____ →

and then this machine _____ →

NOTE: If you think that you can get your answer by using
fewer than 3 machines then write nothing on the lines you do not need.

9. Your starting number is _____ → 3

If you wish to change this number into _____ → 10

you will use this machine _____ →

and then this machine _____ →

and then this machine _____ →

NOTE: If you think that you can get your answer by using
fewer than 3 machines then write nothing on the lines you do not need.

One week after writing the Initial Learning Tests subjects were given one of the Transfer Tests after a brief introduction which is printed at the beginning of each of these tests. This time they were allowed to use the Machine Description and Machine Reminder charts, but not the Initial Teaching Booklet for Game B. For the other Game in the Transfer Test they were provided with the Machine Description and in most tests also with the Machine Reminder Charts.

5. The Transfer Tests.

In each of the Transfer Tests a pair of representations was presented to subjects simultaneously. Every question would demand the completion of a task in one representation and then immediately also in the other representation. The order in which the representations appeared was reversed between any one question and the next. To achieve balance questions were matched in pairs according to the number of decision points in their tasks. Table 4 - 4 contains a sample of a pair of matched questions from the Transfer Test¹ between Game B and Game C. In question 9 subjects had first to find in their correct order the Machines which took them from the first to the second of the given frames in Game C. Then they had to find the corresponding Machines in Game B, enter their names on the last 4 empty lines in the question and use them to find the number into

1 All Transfer Tests involving two representations of the same type are constructed on identical principles. The Introduction and Task Analysis of this Test will be found in Special Appendices S4 - 3 and S4 - 4 respectively.

Table 4 - 4

A Matched Pair of sample questions from the Transfer Test between Game B and Game C.

9. If in Game C you go from this frame

to this frame

by using this machine

and then this machine

and then this machine

and then this machine

then in Game B you go from this number

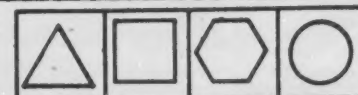
to this number

by using this machine

and then this machine

and then this machine

and then this machine



6

Table 4 - 4 (continued)

10. If in Game B you go from this number
to this number
by using this machine
and then this machine
and then this machine
and then this machine
then in Game C you go from this frame
to this frame
by using this machine
and then this machine
and then this machine
and then this machine

10

30



which the number 6 is transformed in Game B. In question 10 a task ¹ of the same length had to be completed with the order of the representations reversed.

The scoring system had to reconcile the award of equal marks for fully completed tasks in each representation with the award of a proportion of these marks for partially completed tasks. When the first half of a question was a task with decision points the pupils had to find a correct sequence of 2, 3 or 4 Machines. The number of Machines was also the maximum possible score for the task. In the translation of this task pupils applied the mates ² of these Machines in the other representation and also found the final state of the tetrahedron. This gave one additional subdivision of the maximum score. A fractional mark for each correct step resulted and the aggregate of these fractional marks was rounded off to the nearest integer.

The Transfer Test between Game B and Game C is a test within the same type of representation, namely type (3,2). An inspection of the Machine Reminder charts ³ for these two Games reveals that for corresponding Machines the transformation lines with their arrows appear in the same position on the paper in each representation. In addition the Introduction to the Transfer Test so matches up one of the numbers with one of the frames that all numbers and

-
- 1 The task in the second half of each of these questions was thus a task without decision points. In the first four questions of the test the tasks in both halves were without decision points.
 - 2 Finding the sequence of these Machines was easy because the necessary correspondences had been given in the instructions to each Transfer Test.
 - 3 Appendices S3 - 2 and S3 - 3.

frames also correspond in their position on the paper. If a subject became aware of this physical correspondence¹ he would be able to find the answer to the second half of each question in the Transfer Test just from a superimposition of the charts, without thinking of the function of the Machines in the second representation at all. To avoid mistaking a transfer made in this manner for genuine understanding of the representation the frames in the charts for Game C were permuted from their original position before being used in the test,² and in consequence some of the transformation lines also assumed new positions. One chart for Game C in its original and its permuted form is given in Table 4 - 5, and Table 4 - 6 lists the permutations to which the rows and columns of the charts for Game A, Game C and Game D were subjected.³ The charts for Game B remained in their original form throughout, since physical correspondence was already broken by the other three permutations. No transfer tests between two representations of type (3,3) were constructed, and no permutations were therefore applied to the charts for these representations. In any transfer test across the two types of representation physical correspondence was broken by a change in the order of one of the machines and by the avoidance of transfer

1 Chapter 1, p. 54.

2 There was one exception to this practice. See Chapter 5, p.136.

3 All permuted charts will be found in Special Appendix S4 - 5.

Table 4 - 5a.

Original Form of one Machine Reminder Chart for Game C.

The SWITCH MACHINE moves the shapes around like this

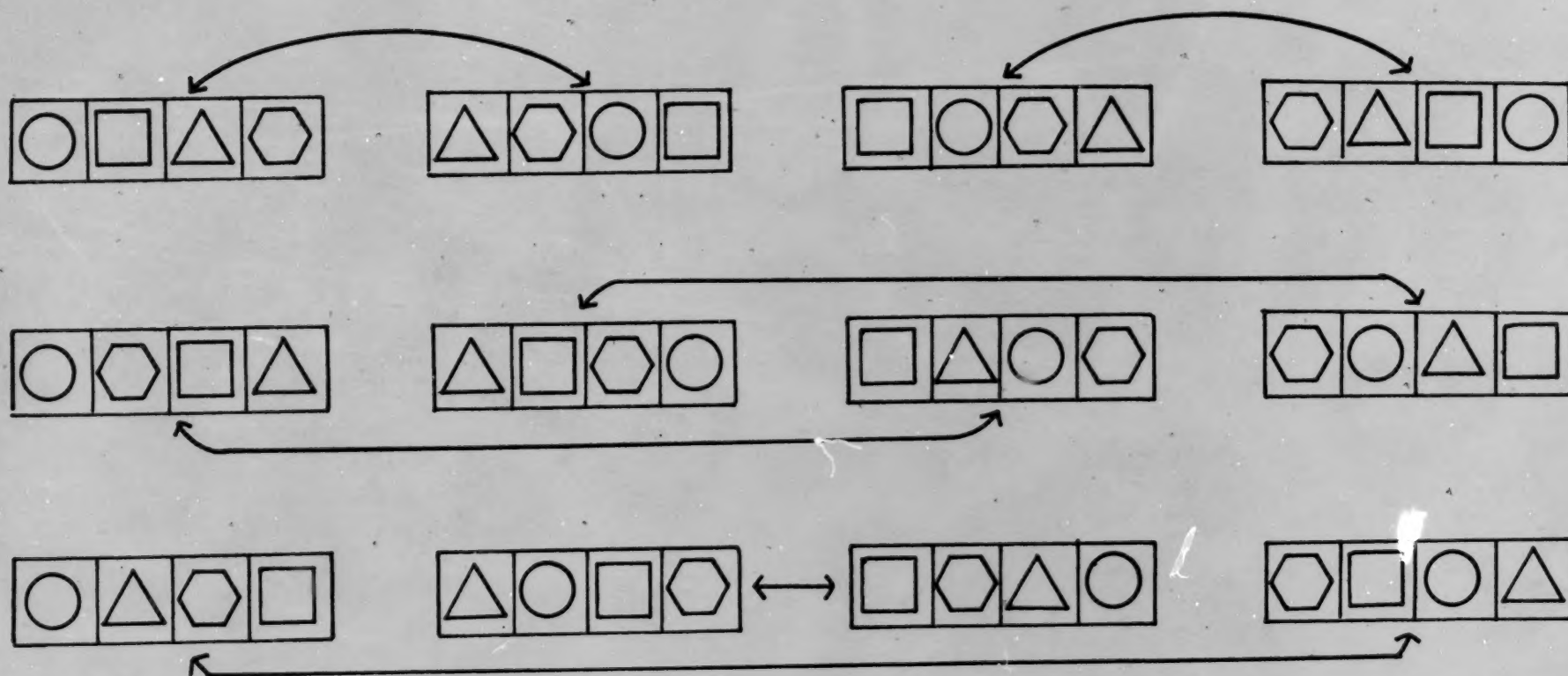


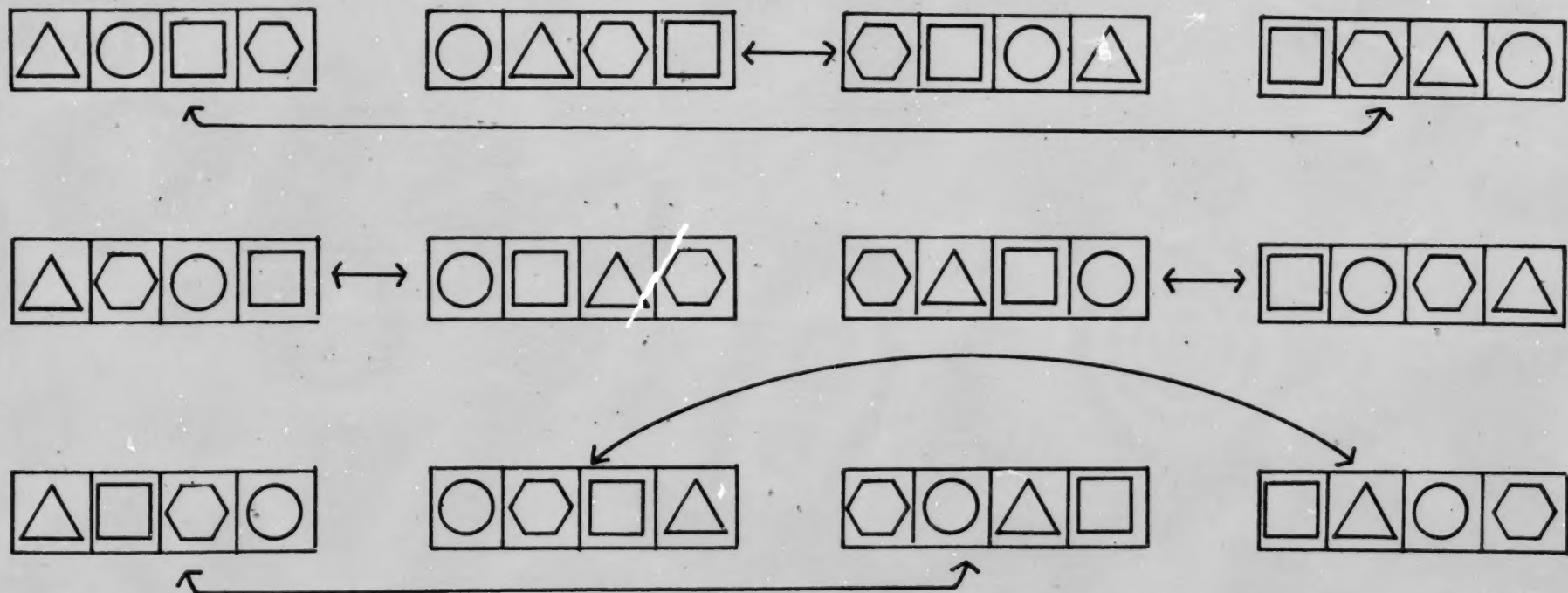
Table 4 - 5b.

Permuted Form of the same Machine Reminder Chart for Game C.

Row Perm (123)

Col Perm (12)(34)

The SWITCH MACHINE moves the shapes around like this



tasks which required only the use of the machine that was common to both representations.

Table 4 - 6

The Permutations assigned to the Machine Reminder
Charts of the four Games of Type (3,2).

Game	Row Permutation	Column Permutation
A	(1)(23)	(1)(234)
B	none	none
C	(123)	(12)(34)
D	(123)	(14)(23)

In the construction of the transfer tests between a type (3,2) and a type (3,3) representation two new features presented themselves. In the first place the only common elements in any transfer task were now the starting and the final position, since a change in the order of one of the machines no longer permitted the matching of the machines in the two representations on a one:one basis in any sequence of transformations. The second feature, which arose in consequence of the first, was that tasks in the two halves of a given question were not always of the same length. This is evident from Table 4 - 7. Because of the first feature all tasks in both halves of each question now became

Table 4 - 7

The Equivalence of Tasks in Game A and Game E.

Game A		Game E	
Task X^3 or Y^2	No. of Steps 3 or 2 or 0	No of Steps 3 or 0	Task X^3 or Z^3
X	1	1	X
X^2	2	2	X^2
Y	1	2	XZ
XY	2	3	X^2Z
YX	2	2	Z^2
X^2Y	3	1	Z
YX^2	3	3	Z^2X
XYX	3	3	XZ^2
YXY	3	3	ZX^2
X^2YX	4	2	ZX
XYX^2	4	4	ZX^2Z

All the tasks in the first row are improper in that they are equivalent to the identity transformation.

tasks with decision points,¹ and because of their unequal length resulting from the second feature the scoring system required some modification.

The equivalence of tasks between Game A and Game E as given in Table 4 - 7 holds for all Transfer Tests between representations of different type. There is only one 4 step task available in Game E, and only two of the 2 step tasks correspond in the two representations. The Transfer Test² between Game B and Game F was constructed with a view to preserving the pattern of maximum scores as set by the transfer tests between representations of the same type. One of the Machines, namely the Two-four Machine, remained the same in both representations. Since the stimulus material was also identical no task involving only the use of this Machine could be incorporated into the test. As in the case of the transfer tests between representations of the same type the introduction of an arbitrary element into the scoring system could not be avoided, because pairs of tasks in one and the same question could not always be matched on the number of steps required for their completion. If the pattern of maximum scores required a maximum mark of 2 for a particular task and a 2 step task was not available in one of the representations, then a 3 step task was given together

1 Because the Machine sequence of the second representation could not be obtained from the sequence in the first representation in any simple way, the initial and final states provided the most obvious connection between the two representations.

2 Appendices S4 - 6 and S4 - 7.

with the Machine required for the first step. Question 9 required a 4 step task in both representations. The only 4 step task available in representations of type (3,3) was used in question 10. Therefore in question 9 the 4 step task in Game B was matched with a 2 step task in Game F, but the number of steps required was not ¹ directly given by the number of lines provided for the answer. The mark for the 2 step task was only allowed to assume the values 0, 2 or 4.

Table 4 - 8 contains two sample questions from this test. Since the Two-four Machine served as common machine in both representations, subjects were only given three Machine Reminder charts instead of the usual four.

The eight representations ² of Chapter 3 could yield $8 \times 7 = 56$ transfer tests. To this must be added the possibility of withholding in each test the charts for none, one, or both representations, also the possibility of using different permutations of the charts when they are used in both representations. The seven transfer tests from this large collection which were actually administered are listed in Table 4 - 9.

1 General Appendix G6.

2 p. 101.

Table 4 - 8

A Matched Pair of Sample Questions from the Transfer Test between Game B and Game F.

3. If in Game B you go from this number	→	3
to this number	→	2
by using this machine	→	_____
and then this machine	→	_____
then in Game F you go from this number	→	3
to this number	→	2
by using this machine	→	_____
and then this machine	→	_____

(continuation)

Table 4 - 8

4.	If in Game F you go from this number	—————→	5
	to this number	—————→	2
	by using this machine	—————→	TWO-FOUR
	and then this machine	—————→	—————
	and then this machine	—————→	—————
	then in Game B you go from this number	—————→	5
	to this number	—————→	2
	by using this machine	—————→	—————
	and then this machine	—————→	—————

Table 4 - 9.

The seven Transfer Tests used in the experiments.

Test No.	Between Games	Charts available for Game(s)	Charts permuted for Game
1	B and C	B and C	C
2	B and C	B and C	None
3	B and C	B	None
4	B and A	B and A	A
5	B and D	B and D	D
6	B and F	B and F	None
7	B and G	B and G	None

6. The Scoring System of the Transfer Tests.¹

In each of these tests the direction of the transfer between representations changes from one question to the next. The questions are matched² in pairs on the number of steps which their tasks require for completion. Let us call the task from which the transfer is made the leading task, and the task to which the transfer is made the trailing task. In all cases the maximum number of points obtainable for the leading and the trailing half of a question is the same. Since questions are matched in pairs any one such pair will yield 4 scores whose maximum

1 All the Transfer Tests will be found in the General Appendices G3 to G7.

2 Between representations of different type this matching was achieved through the modifications described on pp. 121 - 122.

is equal. With a total of 10 questions in the test this produces a matrix which will be referred to as the Equivalence Matrix. It consists of 5 rows corresponding to the 5 pairs of matched questions, and of 4 columns in which the questions of each pair are divided into a trailing and a leading half.

The representations are assigned to the columns as follows:

Column 1	Old representation leading
Column 2	New representation trailing
Column 3	New representation leading
Column 4	Old representation trailing

The old representation is the representation learnt from the Initial Teaching Booklet, and the new representation the one encountered in the transfer test for the first time. Table 4 - 10 shows the Equivalence Matrix with its 20 entries at maximum, and in Table 4 - 11 the scores obtained by a hypothetical subject have been inserted into the same matrix.

Table 4 - 10

The Equivalence Matrix

Columns Questions	1	2	3	4
1 & 2	1	1	1	1
3 & 4	2	2	2	2
5 & 6	2	2	2	2
7 & 8	3	3	3	3
9 & 10	4	4	4	4
Totals	12	12	12	12

Table 4 - 11

The Equivalence Matrix of a Hypothetical Subject

Columns Questions	1	2	3	4
1 & 2	1	0	1	1
3 & 4	2	1	1	1
5 & 6	2	2	0	0
7 & 8	0	0	3	2
9 & 10	4	2	4	1
Totals	9	5	9	5

From every such Equivalence Matrix it is possible to calculate the 16 scores as described and codified in Table 4 - 12. For the matrix of Table 4 - 11 the values of these scores would be as follows :

(1) C_1	= 9	(9) MAX_{12}	= 9
(2) C_2	= 5	(10) D_{12}	= 4
(3) C_3	= 9	(11) MIN_{34}	= 5
(4) C_4	= 5	(12) MAX_{34}	= 9
(5) MIN_{1234}	= 2	(13) D_{34}	= 4
(6) MAX_{1234}	= 12	(14) MIN_{13}	= 6
(7) D_{1234}	= 10	(15) MAX_{13}	= 12
(8) MIN_{12}	= 5	(16) D_{13}	= 6

Table 4 - 12

The Sixteen Measures obtainable from the Equivalence Matrix.

Description	Code
(1) The sum of the scores in column 1	C ₁
(2) The sum of the scores in column 2	C ₂
(3) The sum of the scores in column 3	C ₃
(4) The sum of the scores in column 4	C ₄
(5) The sum of the smallest scores occurring in each row	MIN ₁₂₃₄
(6) The sum of the largest scores occurring in each row	MAX ₁₂₃₄
(7) The difference between the last two sums	D ₁₂₃₄
(8) The sum of the smallest scores occurring in each row of the submatrix formed by columns 1 and 2 only	MIN ₁₂
(9) The sum of the largest scores occurring in each row of the same submatrix	MAX ₁₂
(10) The difference between the last two sums	D ₁₂
(11) The sum of the smallest scores occurring in each row of the submatrix formed by columns 3 and 4 only	MIN ₃₄
(12) The sum of the largest scores occurring in each row of the same submatrix	MAX ₃₄

Table 4 - 12 continued

Description	Code
(13) The difference between the last two sums	D_{34}
(14) The sum of the smallest scores occurring in each row of the submatrix formed by columns 1 and 3 only	MIN_{13}
(15) The sum of the largest scores occurring in each row of the same submatrix	MAX_{13}
(16) The difference between the last two sums	D_{13}

Scores for measures with suffixes of four figures are computed from the whole matrix and scores for measures with two suffixes from the submatrix of the columns indicated by the figures. There are therefore three partial and one total score to be computed from each of the MIN, MAX and D measures. Two of the partial scores are obtained from the translation of tasks in both directions, and the third from comparable tasks unrelated by translation.

The partial MIN scores are computed by summing the lesser of the two scores that occur in the intersection of each row with the two columns indicated by the suffixes. They thus provide a more accurate measure of the amount of

performance that is common to the two representations than would the lesser of the two column totals, since a subject will not necessarily produce the lower score in the same column in each task.¹ The difference between the partial MIN score and the lower of the column totals then measures the degree to which a subject's low scores oscillate between the two columns. If the difference is 3 points and the lower of the column totals is 8 then the subject's low score has occurred in the other column 3 times out of 8. By a natural extension of this principle the total MIN score is computed from the least of all the four scores in each pair of matched tasks. If there is a difference between the total MIN score and the lowest of the partial MIN scores then this again indicates that a portion of the subject's low performance is to be found in one or both of the columns on which the partial MIN score in question does not depend. The total MIN score thus measures the amount of performance that is equally distributed over the four columns - the degree to which the two representations are psychologically equivalent.

The function of the four MAX scores is antipodal, as it were, to that of the four MIN scores. Each measures the subject's best performance in two or all four columns, without regard to the representation in which this performance

1 Even in cases where the task in the second column is wholly dependent on the translation of the task in the first column it is possible for a subject to understand this task, complete it incorrectly, and then translate it correctly into the representation of the second column. Instances of this occurred occasionally during the present experiments.

occurs. The high scores, like the low scores, may oscillate. In that event one could conclude that the subject was gaining through the use of two representations in the transfer test. Proof of this gain necessitates some psychological inequivalence, but the mere presence of the latter is no proof of gain, since a subject's best performance may be entirely confined to one column.

The MIN scores measure partial and total psychological equivalence. At the same time they also measure a level of performance, the level up to which psychological equivalence is present. It is this second feature which makes it possible to distinguish subjects who perform equally well in both representations from subjects who perform equally badly. A measure of equally distributed ignorance is readily obtained by subtracting the MAX_{1234} score from 12, the maximum possible score for each column in the matrix. This ignorance is a kind of psychological equivalence - a negative kind such as exists when someone is entirely ignorant of two languages. The scoring system as developed here gives precedence to the measure of positive psychological equivalence as being more informative. Even if the MIN_{1234} score were equal to zero this would only mean that no performance whatsoever was equally distributed over the four columns. No indication would have been given as to the amount of ignorance that was equally distributed. But if MAX_{1234} also were zero then the score of the amount of ignorance or negative psychological equivalence would be 12.

Table 4 - 13

Partial orders among the sixteen measures obtainable
from the Equivalence Matrix

1. $\text{MIN}_{1234} \leq \text{the smallest of } (C_1, C_2, C_3, C_4) \leq \text{the largest of } (C_1, C_2, C_3, C_4) \leq \text{MAX}_{1234}.$
2. $\text{MIN}_{12} \leq \text{the smallest of } (C_1, C_2) \leq \text{the largest of } (C_1, C_2) \leq \text{MAX}_{12},$ and two similar inequalities relating MIN_{34} with MAX_{34} , and MIN_{13} with $\text{MAX}_{13}.$
3. $\text{MIN}_{12} \geq \text{MIN}_{1234},$ and two similar inequalities for MIN_{34} and $\text{MIN}_{13}.$
4. $\text{MAX}_{12} \leq \text{MAX}_{1234},$ and two similar inequalities for MAX_{34} and $\text{MAX}_{13}.$

Certain partial orders which exist between the C, MIN and MAX measures in virtue of their definition are listed in Table 4 - 13. These orders impose restrictions on the testing of some significant differences, for instance between MIN_{12} and MIN_{1234} . The first of these can never be less than the second, but the hypothesis alternative to the hypothesis of statistical equality is formulated on the assumption that inequality in both directions is possible.

The D scores, which are computationally dependent on the MIN and MAX scores, provide measures of the discrepancy which exist between the best and lowest common performance in the total matrix or one of the three submatrices. They could be interpreted as measures of the amount of stress experienced by subjects during transfer to a second representation, as distinct from the stress created by the difficulty of the individual questions as such. The one feature common to positive and negative psychological equivalence is that both are free of stress. The D scores measure the amount of performance that is unequally distributed and thus complete the threefold partition of the matrix and each submatrix into scores of equally distributed performance, equally distributed ignorance, and unequally distributed performance (or ignorance). No measure of the direction of the stress is contained in the D score. In the case of the partial D scores arising from just two columns at a time the stress is easily expressed as the sum of two directional stresses. For example, $D_{12} = MAX_{12} - MIN_{12} = (MAX_{12} - C_1) + (C_1 - MIN_{12})$.

$(MAX_{12} - C_1)$ measures the amount gained over the C_1 score in the two columns when the higher of the two entries is counted in each row. It represents the gain in column 2 over column 1. $(C_1 - MIN_{12})$, on the other hand, measures the amount lost to the C_1 score when the lower of the two entries is counted in each row. This amount is always equal ¹ to the amount gained over the C_2 score. In the case of the total stress score D_{1234} the situation is more complicated, since any point that is not gained in one of the four columns may give rise to stress by being gained in one, two or three of the other columns. Thus in any unequally distributed performance the number of points lost may equal, exceed or fall short of the number of points gained.

In this way then the measure of psychological equivalence together with all its associated measures forms part of an integrated whole. Each measure provides specific information which any comprehensive investigation of psychological equivalence must take into account. To explore the interrelated functions of all the measures in applications will be the concern of the chapter that follows.

1 Since $MIN_{12} + MAX_{12} = C_1 + C_2$.

CHAPTER 5

THE EXPERIMENTS AND THEIR ANALYSIS

The 7 Transfer Tests ¹ were administered to 8 Treatment Groups of students in Junior High and Senior High Schools ² as shown in Table 5 - 1. All students took their Initial Learning Test in Game B, which was therefore the familiar representation in every Transfer Test. The Transfer Test between Game B and Game C with permuted Machine Reminder Charts for Game C was given to three grades at both the Junior and Senior levels. All other Transfer Tests were given to three grades at just one or the other of these levels. The Transfer Test between Game B and Game C without permutation of the Machine Reminder Charts was administered at the Junior High School level only, whereas the same test without Machine Reminder Charts for Game C was only administered at the Senior High School level. Of the remaining Transfer Tests that between Game B and Game D was administered at the Junior High level and the others at the Senior High level, the assumption being that Game A in symbolic form as well as Games F and G which contained changes in the Type ³ of the representation would be more difficult and therefore better suited for older students.

1 Chapter 4, p. 125.

2 All the schools were under the direction of the School Boards of Halifax and Dartmouth, Nova Scotia, Canada. Junior High Schools (grades 7, 8, 9) and Senior High Schools (grades 10, 11, 12) are generally independent educational units housed in separate buildings.

3 Chapter 3, p. 101.

Besides the analyses of the 7 Transfer Tests the present Chapter also contains an account of a subsidiary experiment¹ directed at the investigation of the transitivity of relationships when transfer between three representations is tested. In the special tests designed for this purpose tasks from Game B, Game C and Game D were used, but the increase in the number of representations from two to three necessitated a reduction of the mathematical content that could be sampled.

The Chapter is divided into five Sections:

- A. All Treatment Groups in Combination.
- B. Comparisons between Subgroups selected from the Total Group on the Basis of Special Score Characteristics.
- C. Comparisons between the 8 Treatment Groups.
- D. Contributions of the Test Items to the Score Characteristics.
- E. The Determination of a Distance Measure.

In Section A some important characteristics of the scores supplied by the Total Group of pupils tested are analysed. In Section B Subgroups are created on the basis of some of these characteristics and compared with one another. In Section C the 8 Treatment Groups are compared on all the scores directly obtainable from the Equivalence Matrix. In Section D changes in performance are shown to be associated with changes in the items as the pupils progress through their tests. Finally in Section E a single measure is constructed to represent differences between the performances of the same subject in two representations, and then examined for its possession of the properties of a metric.

1 The subjects used in the subsidiary experiment were Junior High School students in St. John's, Newfoundland.

Table 5 - 1

The 8 Treatment Groups.

Treatment Group No.	Transfer Test No.	Between Games	Charts available for Game(s)	Charts permuted for Game	Grades	No. of subjects
1	1	B and C	B and C	C	7, 8, 9	86
2	1	B and C	B and C	C	10, 11, 12	39
3	2	B and C	B and C	none	7, 8, 9	64
4	3	B and C	B	none	10, 11, 12	54
5	4	B and A	B and A	A	10, 11, 12	51
6	5	B and D	B and D	D	7, 8, 9	67
7	6	B and F	B and F	none	10, 11, 12	54
8	7	B and G	B and G	none	10, 11, 12	45
						Total 460

Section A.

All Treatment Groups in Combination.

Table 5 - 2

The Mean Scores of the Total Group

460 subjects $IQ^1 = 111.5$. $IL = 16.8$.				
Columns	C_1 9.7	C_2 7.6	C_3 9.2	C_4 7.7
Equivalence	MIN_{1234} 6.2	MIN_{12} 7.4	MIN_{34} 7.5	MIN_{13} 8.5
Best Performance	MAX_{1234} 10.6	MAX_{12} 9.9	MAX_{34} 9.5	MAX_{13} 10.5
Stress	D_{1234} 4.4	D_{12} 2.5	D_{34} 2.0	D_{13} 2.0

The C_2 means were considerably lower than the C_1 means and a similar difference is noticeable between the C_4 and the C_3 means. Wilcoxon² tests between both sets of scores and also between the C_2 and C_3 scores revealed highly significant

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- 1 The abbreviations IQ and IL will be used throughout for intelligence and for performance on the Initial Learning Test respectively. The maximum possible score for the latter test was 20.
 - 2 Owing to the lack of knowledge about the distribution of the new sets of scores non-parametric tests of significance were used throughout the analysis. The Wilcoxon test is described in Siegel (1956), pp.75 - 83.

differences (Wcx. $p = 0.000$ in all cases).¹ The C_1 and C_3 mean scores were closer together, but the differences between these two sets of scores nevertheless remained highly significant (Wcx. $p = 0.000$). There was thus much discrepancy between individual subjects, but the reversals in the direction of this discrepancy tended to cancel each other out in the calculation of the mean scores. Table 5 - 3 shows that only a very small number of subjects scored higher in C_2 than in C_1 , or higher in C_4 than in C_3 , but that the directions of the inequalities between C_1 and C_3 were more evenly distributed.²

Table 5 - 3

The Frequency Distributions of the Equalities and Inequalities between the four Column Scores.

No. Ss 261	$C_1 > C_2$	No. Ss 185	$C_1 = C_2$	No. Ss 14	$C_1 < C_2$
209	$C_3 > C_4$	218	$C_3 = C_4$	33	$C_3 < C_4$
150	$C_1 > C_3$	212	$C_1 = C_3$	98	$C_1 < C_3$

- 1 Two-tailed tests were used throughout the analysis and the rejection level was set at 0.05. Most of the comparisons in Sections A and B involved several tests of significance. Since each test of significance increases the risk of incorrectly judging a p value to be significant and thereby committing a Type I Error, the accumulated Type I Error was estimated for all the p values observed below the level of rejection, and found to be less than this level. The method of estimation is explained in Appendix S5 - 2, p.296, in connection with the multiple comparisons of Section C.
- 2 10 of the subjects who scored higher in C_2 than in C_1 and 14 of the subjects who scored higher in C_4 than in C_3 came from the Transfer Tests between representations of different types. In these tests, it will be recalled (Chapter 4, p.119), matching of the individual steps in each task was not possible. In tests involving representations of the same type a higher score could in most of the questions only arise if subjects understood the task in the first representation but failed to express it correctly in that representation.

The difference between the C_2 and the C_4 scores was near to achieving statistical significance (Wcx. $p = .074$), and the same was true of the difference between the MIN_{12} and MIN_{34} scores (Wcx. $p = .068$)¹. The small difference between the C_2 and MIN_{12} means and again between the C_4 and MIN_{34} means shows that subjects produced their low scores predominantly in the second and fourth columns, as already suggested by the low means of the C_2 and C_4 scores themselves. The mean of the total equivalence score MIN_{1234} was considerably lower than either the MIN_{12} or MIN_{34} mean. This indicates that the low scores, which occurred predominantly in the second and fourth columns, in their turn oscillated between these columns from one task to the next. Between $(MIN_{12} - MIN_{1234})$ and $(MIN_{34} - MIN_{1234})$ there was, however, no significant difference (Wcx. $p = .62$), so that these oscillations did not favour either half of the Equivalence Matrix.

The three partial MAX scores MAX_{12} , MAX_{34} and MAX_{13} all differed very significantly (Wcx. $p = .000$). The difference between MAX_{12} and MAX_{34} corresponds to the difference between the C_1 and C_3 totals, in which the high scores were concentrated. MAX_{13} differed from both the others because it was calculated from the better of the first and third column score in each task. For the same reason there was also little difference between the MAX_{13} and MAX_{1234} means.²

-
- 1 Differences between these two partial MIN scores and the MIN_{13} score were highly significant (Wcx. $p = .000$). This accords with what has already been said about the higher level of performance in columns 1 and 3.
 - 2 No test of significance is possible between these two scores, since MAX_{1234} can never be less than MAX_{13} . See Table 4 - 13, Chapter 4, p.133.

The means of the differences ($MAX_{1234} - MAX_{12}$) and ($MAX_{1234} - MAX_{34}$) again show that the high scores, like the low scores, oscillated between the two halves of the Equivalence Matrix. But this time the left half was favoured by a test of significance ($Wcx. p = .001$), since in this half the good scores came from the first column, in which subjects initiated tasks with which they were already familiar from the Initial Learning Test.

This situation is also reflected in the relative standing of the three partial stress scores D_{12} , D_{34} and D_{13} . D_{12} was significantly greater than D_{34} ($Wcx. p = .001$) and also significantly greater than D_{13} ($Wcx. p = .009$), but D_{34} and D_{13} were statistically equal ($Wcx. p = .500$). These results may be explained by the fact that the D_{12} score arose from a performance which involved a double difficulty. In the first place the task had to be dissociated from the representation in which it had been completed in column 1 and re-associated with the representation in column 2. Secondly the representation in column 2 was the unfamiliar one. Between columns 1 and 3 and also between columns 3 and 4 these difficulties were only encountered one at a time.¹

The D_{1234} score, which has been interpreted as the measure of the total stress existing between the two representations, is calculated from the difference between MAX_{1234} , the score of best performance, and MIN_{1234} , the score of equivalence. Now $4 MIN_{1234}$ measures the amount of successful

1 For the subgroup of subjects whose C_1 and C_3 scores were equal the D_{12} and D_{34} scores no longer differed significantly ($Wcx. p = .24$). See also p. 151 *infra*.

performance that is evenly distributed over the four columns, and $4(12 - \text{MAX}_{1234})^1$ the amount of unsuccessful performance or ignorance that is so distributed.² $4 D_{1234}$ therefore³ measures the amount of both successful and unsuccessful performance that is unevenly distributed over the four columns. Successful performance that is unevenly distributed is measured by $(C_1 + C_2 + C_3 + C_4) - 4 \text{MIN}_{1234}$ and unsuccessful performance so distributed by $4 \text{MAX}_{1234} - (C_1 + C_2 + C_3 + C_4)$. The mean of these last two measures is $2 D_{1234}$. Now in every row of the equivalence matrix that contributes to unevenly distributed performance at least one point must be gained and at least one point must be lost. From this it follows that these two measures can never be less than $1 D_{1234}$ and never greater than $3 D_{1234}$.

The total group obtained a mean of 9.2 for unevenly distributed successful performance as against a mean of only 8.2 for unevenly distributed unsuccessful performance. There were highly significant differences between the two sets of scores and also between each of them and $2 D_{1234}$, their mean (Wcx. $p = .000$ in all cases). The frequencies with which each of the three possible order relations between the two types of score occurred are listed in Table 5 - 4.

-
- 1 12 was the maximum possible score in any one column of each test.
 - 2 This then is also a measure of equivalence, but clearly not one that is of primary interest. The present procedures made it possible to keep the two types of equivalence apart.
 - 3 The total number of points in each transfer test was 48. Subtraction of the two types of evenly distributed performance from this number leaves $4 (\text{MAX}_{1234} - \text{MIN}_{1234})$, which is equal to $4 D_{1234}$.

Table 5 - 4

The Frequency Distributions of the Order Relations
between the two Types of Unevenly Distributed Performance.

No. Ss	
115	$4 \text{ MAX}_{1234} - (C_1 + C_2 + C_3 + C_4) > (C_1 + C_2 + C_3 + C_4) - 4 \text{ MIN}_{1234}$
224	$4 \text{ MAX}_{1234} - (C_1 + C_2 + C_3 + C_4) < (C_1 + C_2 + C_3 + C_4) - 4 \text{ MIN}_{1234}$
121	$4 \text{ MAX}_{1234} - (C_1 + C_2 + C_3 + C_4) = (C_1 + C_2 + C_3 + C_4) - 4 \text{ MIN}_{1234}$

Of the 121 subjects who obtained equality only 42 had a D_{1234} score that was greater than zero. The dominance of the successful performance accords with the fact that the total group obtained a much higher ¹ mean score for MIN_{1234} , the measure of evenly distributed successful performance, than for $4 (12 - \text{MAX}_{1234})$, the measure of evenly distributed unsuccessful performance.

It has already been stated² that there was considerable oscillation of the subjects' successful performance between column 1 and column 3, with the result that the MAX_{1234} and MAX_{13} means were higher than those of C_1 and C_3 . Many subjects were able to complete more tasks when offered two representations than they could have completed if they had only been able to work in the familiar representation of Game B and accumulate scores only in column 1. At the same time many subjects could not keep up with their performance in this column in the other

1 See Table 5 - 2 on p. 139.

2 p. 140 supra.

three columns, least of all in columns 2 and 4. Table 5 - 5 lists the frequency distribution of the gains and losses relative to C_1 for all those subjects who could gain on their C_1 score because it stood at less than the maximum of 12. Table 5 - 6 gives the corresponding distribution for the submatrix consisting of columns 1 and 3 only.

Table 5 - 5

The Frequency Distributions of the Order Relations between the Gains and Losses relative to the C_1 Score ($C_1 < 12$).

No. Ss	Gain	Loss
61	$MAX_{1234} - C_1 > C_1 - MIN_{1234}$	
138	$MAX_{1234} - C_1 < C_1 - MIN_{1234}$	
18	$MAX_{1234} - C_1 = C_1 - MIN_{1234}$	

Table 5 - 6

The Corresponding Distribution for the Submatrix consisting of Columns 1 and 3 only.¹

No. Ss	Gain	Loss
98	$MAX_{13} - C_1 > C_1 - MIN_{13}$	
74	$MAX_{13} - C_1 < C_1 - MIN_{13}$	
45	$MAX_{13} - C_1 = C_1 - MIN_{13}$	

¹ Since the submatrix has only two columns, $C_1 - MIN_{13}$ is equal to $MAX_{13} - C_3$. In other words, a gain relative to C_3 is a loss relative to C_1 , and vice versa. 162

Out of the total number of 217 subjects 61 gained more than they lost over the whole matrix, and of these 22 lost nothing relative to their performance in C_1 . Over the submatrix 98 gained more than they lost, and of these as many as 75 lost nothing. Because of the difficulties caused by dissociation and re-association between C_1 and C_2 and also between C_3 and C_4 the losses far outweighed the gains when the whole matrix was taken into account. But in the submatrix the gains balanced the losses (Wcx. $p = .14$).

Eight hypotheses¹, which suggested themselves during the analyses of Section A, are summarised in Table 5 - 7.

Table 5 - 7

The Hypotheses suggested by the Results of Section A.

Content	Evidence
A1. The translation of a task from one representation into another results in a loss of performance, irrespective of the direction in which the translation proceeds.	(i) The C_1 scores were significantly higher than the C_2 scores. (ii) The C_3 scores were significantly higher than the C_4 scores
A2. The amount of performance lost during translation is independent of this direction, provided the initiation of new tasks is equally good in both representations.	There was no significant difference between the D_{12} and D_{34} scores of those subjects whose C_1 and C_3 scores were equal.

(Continued on next page.)

- 1 Since all the procedures relating to the transfer tests were developed and used for the first time in the present experiments all the hypotheses formulated should be regarded as 'a posteriori' rather than as 'a priori'. All of them suggested themselves only after some experience with the equivalence matrix and the scoring system had been gained, and most of them were formulated only after the relevant scores had been inspected.

Continuation of Table 5 - 7

Content	Evidence
A3. Starting a new task in a new representation is easier than translating a task into the new representation after it has been completed in the familiar representation.	(i) The D_{12} scores were significantly greater than the D_{13} scores. (ii) The C_3 scores were significantly greater than the C_2 scores. (iii) The number of subjects with $C_3 = C_1$ or $C_3 > C_1$ was much greater than the number with $C_2 = C_1$ or $C_2 > C_1$.
A4. In general translating a task back into the familiar representation after it has been completed in the new representation is easier than translation in the other direction, but more difficult than starting a new task in a new representation.	(i) The D_{12} scores were significantly greater than the D_{34} scores for the total group. (ii) The number of subjects with $C_3 = C_1$ or $C_3 > C_1$ was much greater than the number with $C_4 = C_3$ or $C_4 > C_3$. (iii) The C_3 scores were significantly greater than the C_4 scores.
A5. The low scores accumulated in the equivalence matrix oscillate between columns 2 and 4 from one task to the next. This oscillation does not favour either half of the equivalence matrix.	There was no significant difference between the $(MIN_{12} - MIN_{1234})$ and the $(MIN_{34} - MIN_{1234})$ scores.
A6. The high scores oscillate between columns 1 and 3 in a similar manner, but their oscillation favours that half of the equivalence matrix in which tasks are initiated in the familiar representation.	The $(MAX_{1234} - MAX_{12})$ and the $(MAX_{1234} - MAX_{34})$ scores differed significantly.

Continuation of Table 5 - 7

Content	Evidence
A7. The amount of successful performance that is unevenly distributed over the equivalence matrix tends to be greater than the amount of unsuccessful performance so distributed. Both amounts differ significantly from $2 D_{1234}$, their mean.	(i) The mean score for $(C_1 + C_2 + C_3 + C_4) - 4 \text{ MIN}_{1234}$ was higher than for $4 \text{ MAX}_{1234} - (C_1 + C_2 + C_3 + C_4)$. (ii) Both these scores differed significantly from $2 D_{1234}$, and also from each other.
A8. If performance in the familiar representation is not at maximum then transfer to a new task in a new representation is as likely to produce a gain in the overall amount of successful performance as it is not.	There was no significant difference between the $(\text{MAX}_{13} - C_1)$ and the $(C_1 - \text{MIN}_{13})$ scores.

Hypotheses A1 to A4 bring out the relative difficulty of the three types of transfer between columns 1 and 2, 1 and 3, 3 and 4. Hypothesis A5 stresses the fact that translation as such rather than the familiarity of the representation is responsible for low performance in the translated tasks. Hypotheses A6 and A8 seem at a first glance to be in conflict with one another, but this inconsistency is resolved by the condition attached to Hypothesis A8. Subjects tended to score higher in C_1 than in C_3 , but if they fell short of the maximum score in C_1 then they were as likely to gain from the easiest type of transfer as they were not. Hypothesis A7 is in keeping with the fact that the mean score for MIN_{1234} , the equally distributed performance was 6.2, whereas the mean score for

equally distributed ignorance was only 1.4¹.

The Section which follows was designed to provide more information about the differential effects of symmetry arising from transfer under the easiest of conditions, when no translation was involved, of gain through transfer under the same conditions, and of changes in the proportion of successful performance within the unequally distributed performance.

Section B.

Comparisons between Subgroups selected from the Total Group on the Basis of Special Score Characteristics.

$$C_1 = C_3 \quad \text{No. 1} \quad \text{compared with } C_1 \neq C_3.$$

Table 5 - 8

Mean Scores of the Subgroup with $C_1 = C_3$ and $MIN_{1234} < 12$.
119 Subjects. IQ = 111. IL = 16.9 .

Columns	C_1 10.1	C_2 7.3	C_3 10.1	C_4 7.8
Equivalence	MIN_{1234} 5.9	MIN_{12} 7.3	MIN_{34} 7.7	MIN_{13} 9.9
Best Performance	MAX_{1234} 10.4	MAX_{12} 10.1	MAX_{34} 10.1	MAX_{13} 10.3
Stress	D_{1234} 4.5	D_{12} 2.8	D_{34} 2.4	D_{13} 0.46

The purpose behind this selection was to study the effects of the presence of some symmetry² in the Equivalence Matrix.

- 1 Table 5 - 2, p. 139. The mean score 1.4 is obtained by subtraction of the mean score for MAX_{1234} from 12.
- 2 A more stringent condition of symmetry would have been $D_{13} = 0$. The present condition was preferred so as not to reduce the number of available subjects unduly.

Table 5 - 9

Mean Scores of the Subgroup with $C_1 \neq C_3$.

248 Subjects. IQ = 109. IL = 16.

Columns	C_1 8.7	C_2 6.0	C_3 7.8	C_4 6.2
Equivalence	MIN_{1234} 4.3	MIN_{12} 5.8	MIN_{34} 5.7	MIN_{13} 6.5
Best Performance	MAX_{1234} 10.1	MAX_{12} 8.9	MAX_{34} 8.2	MAX_{13} 10.0
Stress	D_{1234} 5.9	D_{12} 3.1	D_{34} 2.5	D_{13} 3.6

The mean scores of the symmetric and asymmetric group are listed in Tables 5 - 8 and 5 - 9. In the selection of the symmetric group the restriction $MIN_{1234} < 12$ was imposed because it automatically applied to the asymmetric group. Neither group therefore contained subjects whose scores were at maximum in every column. Inspection of the tables shows that the symmetric group was far ahead of the asymmetric group in all the four C and in all the four MIN scores. Mann Whitney U tests ¹ comparing the two groups yielded highly significant differences (Mwu p = .000) in all cases.

In the partial stress score ² D_{12} the symmetric group was the better group at translation from the familiar representation and came near to statistical inequality (Mwu p = .055). Within the asymmetric group the D_{12}

1 The non-parametric Mann Whitney U test is described in Siegel (1956), pp. 116 - 127.

2 The very low D_{13} mean score of the symmetric group arose from the criterion governing its selection.

score was significantly greater than the D_{34} score (Wcx. $p = .001$), whereas no such difference appeared within the symmetric group¹.

In the MAX_{12} and MAX_{34} scores, which are closely dependent on the C_1 and C_3 scores respectively, highly significant differences again appeared in favour of the symmetric group (Mwu $p = .000$ in both cases). In the MAX_{1234} and MAX_{13} scores, on the other hand, the two groups ceased to diverge significantly (Mwu $p = .221$ and $.170$ respectively). This means that the asymmetric group oscillated in their successful performance between the columns 1 and 3, and in the two scores in which this oscillation is ignored and the best performance is counted in each row regardless of its distribution among the columns they achieved about as much as the symmetric group. In order to do this they had to gain significantly more relative to their C_1 score (Mwu $p = .000$ for the difference ($MAX_{1234} - C_1$)). But their lesser ability to distribute performance over the matrix is again reflected in their significantly higher D_{1234} score (Mwu $p = .000$). If the unevenly distributed performance is further subdivided into successful and unsuccessful performance² the symmetric group obtained a mean of only 6.3 for unsuccessful performance and a mean of 11.7 for successful performance, whereas the asymmetric group obtained corresponding means of 11.7 and 11.5. The higher D_{1234} score of the asymmetric group arose thus almost entirely from unsuccessful performance.

1 Compare Hypothesis A2 on p.146 supra. The present result provides additional support for this hypothesis because all subjects who attained full score were excluded.

2 See p. 143 supra for the scoring of these performances.

No. 2

(a) $(C_1 + C_2 + C_3 + C_4) - 4 \text{ MIN}_{1234}$ greater than
 $\text{MAX}_{1234} - (C_1 + C_2 + C_3 + C_4)$
 compared with

(b) $(C_1 + C_2 + C_3 + C_4) - 4 \text{ MIN}_{1234}$ less than
 $4 \text{ MAX}_{1234} - (C_1 + C_2 + C_3 + C_4)$.

Table 5 - 10

Mean Scores of the Subgroup with Score Characteristic (a).

204 Subjects. IQ = 112.5 . IL = 17.5 .

Columns	C ₁ 10.3	C ₂ 8.0	C ₃ 10.0	C ₄ 8.3
Equivalence	MIN ₁₂₃₄ 5.9	MIN ₁₂ 7.9	MIN ₃₄ 8.0	MIN ₁₃ 9.4
Best Performance	MAX ₁₂₃₄ 10.9	MAX ₁₂ 10.5	MAX ₃₄ 10.4	MAX ₁₃ 10.9
Stress	D ₁₂₃₄ 5.0	D ₁₂ 2.6	D ₃₄ 2.4	D ₁₃ 1.5

This selection produced two highly contrasted groups, as is evident from Tables 5 - 10 and 5 - 11. The group with score characteristic (a) will be referred to as the successful group because it scored more good points in its unevenly distributed performance than the group with score characteristic (b), which will be referred to as the unsuccessful group. Highly significant differences (Mwu p = .000 throughout) were recorded between the groups in all the 16 scores obtained from the Equivalence Matrix, as well as in the IQ and IL scores. The unsuccessful group were weaker in all respects and

Table 5 - 11

Mean Scores of the Subgroup with Score Characteristics (b).

115 Subjects. IQ = 105. IL = 14.3 .

Columns	C ₁	C ₂	C ₃	C ₄
	7.5	3.5	6.2	3.7
Equivalence	MIN ₁₂₃₄ 2.7	MIN ₁₂ 3.3	MIN ₃₄ 3.3	MIN ₁₃ 4.6
Best Performance	MAX ₁₂₃₄ 9.4	MAX ₁₂ 7.7	MAX ₃₄ 6.5	MAX ₁₃ 9.2
Stress	D ₁₂₃₄ 6.7	D ₁₂ 4.4	D ₃₄ 3.2	D ₁₃ 4.6

experienced more stress as measured by both the total and the partial D scores. Unlike the asymmetric group in the previous comparison they also showed less knowledge of the tasks than their competitors, when in the scoring of this knowledge considerations of its distribution over the matrix were ignored.

One might regard the achievement of the two groups as representing two stages in the progress towards perfect psychological equivalence. The successful group experienced less stress, were ahead in the MAX and MIN scores, and also in the number of good points they scored when their performance was unevenly distributed. This latter performance was thus much nearer to becoming counted as total equivalence than was the unevenly distributed performance of the unsuccessful group.

No. 3

- (a) $MAX_{13} - C_1$ greater than $C_1 - MIN_{13}$ (C_1 less than 12)
compared with
- (b) $MAX_{13} - C_1$ less than $C_1 - MIN_{13}$ (C_1 less than 12).

It was shown in Section A ¹ that within the total group subjects tended to lose more than they gained relative to the familiar representation of Game B in column 1 when their scores in the total Equivalence Matrix were taken into account, and that the gains and losses were balanced over the total group when the scores of columns 2 and 4 were left out of consideration.

The two groups of the present comparison were selected to represent the two possible types of imbalance between loss and gain ² arising from columns and 1 and 3 only. They will be referred to as the gaining and the losing group respectively.

Table 5 - 12

Mean Scores of the Subgroup with Score Characteristic (a).

98 Subjects. IQ = 109. IL = 15.8 .

Columns	C ₁	C ₂	C ₃	C ₄
	7.2	5.8	9.9	7.3
Equivalence	MIN 1234 4.8	MIN 12 5.4	MIN 34 7.1	MIN 13 6.8
Best Performance	MAX 1234 10.4	MAX 12 7.6	MAX 34 10.0	MAX 13 10.3
Stress	D 1234 5.6	D 12 2.3	D 34 2.9	D 13 3.5

1 See p.146 supra.

2 To ensure that all subjects could lose and gain the stipulation that C₁ should be at less than its maximum of 12 was incorporated into the selection. Since all subjects had a score greater than 0 in C₁ no corresponding stipulation was necessary at the other end.

Table 5 - 13

Mean Scores of the Subgroup with Score Characteristic (b).

74 Subjects. IQ = 106. IL = 14.5 .

Columns	C ₁	C ₂	C ₃	C ₄
	7.4	3.9	4.5	3.6
Equivalence	MIN ₁₂₃₄	MIN ₁₂	MIN ₃₄	MIN ₁₃
	2.1	3.7	3.0	4.1
Best Performance	MAX ₁₂₃₄	MAX ₁₂	MAX ₃₄	MAX ₁₃
	8.0	7.5	5.0	7.8
Stress	D ₁₂₃₄	D ₁₂	D ₃₄	D ₁₃
	5.9	3.8	2.0	3.7

The C₁ scores of both groups were statistically equal (Mwu p = .653). The C₃ mean scores of Tables 5 - 12 and 5 - 13 show that there was much asymmetry between column 1 and column 3. As reflected in the D₁₃ scores this asymmetry, however, favoured neither group¹ (Mwu p = .849). As an immediate consequence of the statistical equality between the C₁ scores and the large discrepancy between the C₃ scores the gaining group was markedly superior in the MAX₁₃ and MIN₁₃ scores, as well as in the MAX₁₂₃₄ score, which depended mainly on columns 1 and 3 (Mwu p = .000 in all cases).

The C₂ and D₁₂ scores showed that the gaining group also translated better into the new representation (Mwu p = .000 in both cases), but when it came to translating back into the familiar representation from greatly differing levels of performance in C₃ neither group experienced more of the D₃₄

1 This is in keeping with the balance in gains and losses referred to on p. 146 supra.

stress than the other (Mwu $p = .119$). The MIN_{1234} , MIN_{12} and MIN_{34} scores again greatly favoured the gaining group (Mwu $p = .000$ in all cases), who were also superior in IQ and IL (Mwu $p = .035$ and $.015$ respectively).

The overall stress score D_{1234} measuring the amount of unevenly distributed performance favoured neither group (Mwu $p = .784$), and even when this performance was subdivided into successful and unsuccessful performance no significant differences appeared (Mwu $p = .645$ and $.325$ respectively). The overall superior performance of the gaining group which is reflected in the mean sum of its four column scores thus arose from a significantly higher MIN_{1234} score alone.

Five hypotheses which suggested themselves during the analyses of Section B are summarised in Table 5 - 14.

Table 5 - 14

The Hypotheses suggested by the Results of Section B.

Content	Evidence
B1. Subjects who can complete pairs of matched but independent tasks in each of two representations equally well reach a higher level of psychological equivalence in the whole matrix than subjects who cannot. And they also experience less stress in translating tasks out of the familiar representation.	Significant differences between the MIN_{1234} scores of the two groups in Comparison 1. An almost significant difference between the D_{12} scores.

Continuation of Table 5 - 14

Content	Evidence
B2. Subjects who cannot complete such pairs of tasks are not inferior to subject who can, when they are assessed on the knowledge of content without regard to the distribution of this knowledge over the equivalence matrix.	No significant differences between the MAX₁₂₃₄ scores of the two groups in Comparison 1.
B3. Subjects who score more good points than bad points in the performance that is unevenly distributed over the equivalence matrix are nearer to attaining perfect psychological equivalence than subjects who score more bad points than good points.	Highly significant difference between the two groups of Comparison 2, in all the MIN, MAX, C and D scores.
B4. Subjects who gain relative to their score in the more familiar of two representations when they attempt to complete pairs of matched but independent tasks reach a higher level of psychological equivalence in the whole matrix than subjects who lose.	Highly significant difference between the MIN₁₂₃₄ scores of the two groups in Comparison 3.
B5. Subjects who gain in this way are also better at translating tasks from the familiar into the new representation.	Highly significant difference between the D₁₂ scores in the same Comparison.
B6. Subjects who gain in this way also show more knowledge when the distribution of this knowledge over the equivalence matrix is not taken into account.	Highly significant difference between the MAX₁₂₃₄ scores in the same Comparison.

Continuation of Table 5 - 14

Content	Evidence
B7. Subjects who gain in this way have the same amount of unequally distributed performance as subjects who lose.	No significant difference between the D_{1234} scores in the same Comparison.

Hypotheses B1 and B2 of Comparison No. 1 tell us that symmetry between the independent tasks of columns 1 and 3 makes for higher psychological equivalence but not for higher knowledge, in the measurement of which this symmetry is not taken into account. Also the ability to translate into the new representation seems marginally better when symmetry is present.

The two subgroups in Comparison No. 2 were the most highly contrasted, as Hypothesis B3 shows in summary. Inspection of the unequally distributed performance seems to reveal knowledge and psychological equivalence in evolution; if the amount of successful performance within that performance exceeds the amount of unsuccessful performance, then the subject is nearer to maximum knowledge and perfect psychological equivalence¹.

Hypotheses B4 to B7 of Comparison No. 3 concern themselves with two subgroups² taken from the asymmetric group of Comparison No. 1. Hypotheses B4 to B6 state that the ability

1 Where successful performance exceeds unsuccessful performance there must always be some successful translations.

2 These two subgroups would have formed a partition of the asymmetric group if the criterion $C_1 < 12$ had also been imposed during the selection of the¹ latter.

to gain through independent tasks performed in the new representation leads to better translation into that representation and also to more knowledge as revealed over both representations together. The last of these results is important in that it offers justification for providing the weaker pupil with transfer tasks. In the light of the results of the present Section taken together the performance of a pupil that fell short of perfection might be ranked in increasing order of merit according as it was (i) asymmetric and losing relative to the familiar representation, (ii) asymmetric and gaining relative to the familiar representation, and (iii) symmetric between the independent tasks of columns 1 and 3.

Section C.

Comparisons between the Individual Treatment Groups

All the Treatment Groups¹ had Game B as the familiar representation producing the C_1 score. It was therefore possible to compensate for differences in the IQ, IL and C_1 scores between subjects in the different groups by using only the residuals of the other 15 scores obtained from the Equivalence Matrix after the components that could be predicted from IQ, IL and C_1 had been removed². For all except the D scores the regression³ co-efficients ranged from 0.011 to 0.202 for IL, and from 0.454 to 0.965 for C_1 . The computation of some of these scores such as MAX_{12} depended strongly on the C_1 score, but even when these scores were discounted the regression co-efficients for C_1 still ranged from 0.454 to 0.718. In other words, the knowledge of the familiar representation which subjects retained and applied during the transfer tests exercised the dominant influence on their performance in that test. For the D scores the regression co-efficients ranged from -0.245 to -0.139 for IQ, from -0.212 to -0.139 for IL, and from -0.210 to +0.14 for C_1 . The Treatment Groups

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- 1 The raw mean scores of the Treatment Groups will be found in Appendix S5 - 3.
 - 2 The Spearman correlation co-efficient between IQ and each of IL and C_1 was 0.38, while that between IL and C_1 was 0.47.
 - 3 These co-efficients measure the individual contributions of the IQ, IL and C_1 scores. Co-efficients of linear regression only were¹ used after it was found that the insertion of all second order terms obtainable from IQ, IL and C_1 was making no appreciable difference to the total contribution of these three variables. The co-efficients listed here are standardised beta co-efficients. The residuals used in the comparisons were unstandardised.

have been ranked on their mean residuals for each type of score in Appendix S5 - 1¹.

No. 1 - Age Differences.

The only two Treatment Groups working through identical transfer tests were Groups 1 and 2, composed of subjects from grades 7 - 9 and grades 10 - 12 respectively. No significant² differences between them appeared in any of the 15 residual scores. Inspection of the mean residuals³ suggests that the Junior Group were slightly superior in the C_3 , C_4 , MIN_{34} , MAX_{1234} , MAX_{13} and MAX_{12} scores, and that they also experienced more stress throughout. But with the lowest Mwu p value in all the comparisons equal to 0.147 it seems safe to conclude that age was not a crucial determinant of performance. The approaches adopted in the Initial Learning Test and the transfer tests were sufficiently unsophisticated for the entire age range that had been selected.

No. 2 - Changes in the Machine Reminder Charts of Game C.

Treatment Groups 1, 2, 3 and 4 were all given the Transfer Test between Game B and Game C. For Groups 1 and 2 the Machine Reminder Charts of Game C had the states of the

1 p. 287.

2 See Appendix S5 - 2 on p.290 for estimates of the risk of making a Type I Error if a given p value is judged significant in any of the present comparisons. For a p value as low as 0.003 this risk is already 8% and for a value of 0.007 it is greater than 20%. In view of this, p values were taken as indicative of the amount of contrast present but no judgments as to the significance of this contrast were made for values between 0.005 and 0.010. Such judgments would presumably vary with the nature of the context in which the results were applied.

3 Appendix S5 - 1, p. 287.

tetrahedral group permuted¹ relative to the Charts of Game B. Group 3 had no permutations and could match their charts by direct superposition. Group 4 were given Charts with empty frames² and no transformation lines for Game C, to be completed at their discretion. The mean residuals show Group 3 to be ahead of the other three Groups in all the C, MAX and MIN scores except C₂ and MAX₁₂. A low Mwu p value of 0.009 in favour of Group 3 was observed between this Group and Group 4 in the C₃ score, and lower p values of 0.004 and 0.000 in disfavour of Group 3 appeared in the MAX₁₂ score between this Group and Groups 4 and 1 respectively. Together with the raw mean scores³ this suggests that Group 3 produced unusually high C₃ scores when their C₁ scores and the closely associated MAX₁₂ scores are taken into consideration. The absence of permutations in the Machine Reminder Charts may have helped their adapting to the new representation of Game C unusually well, but there is no corresponding evidence to show that these charts also helped them to translate the tasks of column 3 back into the familiar representation of Game B with the same ease. The withdrawal of the permutations failed to facilitate translation, for which it was primarily intended. This failure could have been due to the difficulties inherent in translation which have been discussed before⁴.

1 Chapter 4, p.116.

2 The starting state corresponding to the number 1 of Game B only, was given in the top left corner of each Machine Reminder for Game C.

3 Appendix S5 - 3, p. 298.

4 p.142 supra.

That the scores of Group 3 were unusual in the total collection of Treatment Groups is suggested by the large number of p values recorded against it in Table E¹ of Appendix S5 - 1, almost half of them at the 0.000 level.

Between Group 4 and Group 1 the lowest Mwu p value recorded for any score was 0.181, and between this Group and Group 2 the lowest p value was 0.692. With many of the residual mean scores² of the three groups close together this would suggest that Group 4 found no special difficulties in developing the representation of Game C for themselves from the rules given at the beginning of the Transfer Test. Subjects apparently availed themselves of the guidance provided by the Machine Reminder Charts to a lesser extent that might be supposed, and were able to think out the moves in each representation without being wholly dependent on them. This supposition might also help to account for the failure on the part of Group 3 to produce a perfect translation without any loss in performance from the superposition of their unpermuted Machine Reminder Charts. In that event their usually high C₃ score must be ascribed to the nature of the iconic representation of Game C as such, rather than to the way in which it was related to Game B in the Transfer Test.

No. 3 - The Symbolic Representation of Game A.

This representation was used by Treatment Group 5 in place of the iconic representation of Game C. The group had the lowest mean residuals³ in all the C and MIN scores, and

1 p. 295 infra.

2 Appendix S5 - 1, p. 287.

3 Appendix S5 - 1, p. 287.

also in the MAX_{34} scores¹. In all the D scores, on the other hand, their mean residuals were the highest of all the 8 Treatment Groups. The strongest contrasts between this group and the four groups using the representation of Game C were obtained in favour of Group 3 in the C_3 , C_4 , MIN_{1234} , MIN_{34} , MIN_{13} and MAX_{34} scores (Mwu $p = 0.000, 0.003, 0.001, 0.002, 0.001$ and 0.000 respectively), and in disfavour of Group 3 in the MAX_{12} score (Mwu $p = 0.000$), which was largely dependent on the representation of Game B. In this score Group 5 was as good as Group 1 (Mwu $p = 0.385$) with the highest mean residual, but in the other six scores whose computation was partly or wholly dependent on the symbolic representation the situation was reversed. These observations² go to suggest that adaptation to the symbolic representation was very difficult, as would have been predicted by those who regard this representation as belonging to the culminating stages of the learning process. In Game A subjects were confronted with an unfamiliar type of algebra, the symbols of which were only invested with meaning by the context of the test.

No. 4 - The Verbal Representation

Treatment Group 6 were given the sentences of Game D in place of the iconic representation of Game C. Their mean residuals³ in the C, MIN and MAX_{34} scores were next to or next but one to those of Group 5. In the MAX_{1234} , MAX_{12} and MAX_{13} scores they were the lowest of all the Treatment

1 The other MAX scores were not unusually low because their level could be maintained by performance in the representation of Game B.

2 Further corroborations may be obtained from the Tables of p values in Appendix S5 - 1, p. 287.

3 Appendix S5 - 1, p. 287.

Groups, indicating that Group 6 performed poorly in Game B as well as in Game D. Of the D residual scores that of D_{12} , arising from translation into the verbal representation, was again next highest after that of Group 5, and the D_{1234} residual was almost as extreme.

The strongest contrasts in disfavour of the verbal representation when set against the iconic representation of Game C were observed between the scores of Group 6 and Group 3, whose performance levels in the MAX_{12} score were virtually equal (Mwu $p = 0.956$). The C_2 , C_3 , MIN_{12} and MAX_{34} scores all yielded p values of 0.000, and the differences in the MIN_{34} , MIN_{13} , MAX_{1234} , MAX_{13} and D_{12} scores were also considerable (p values between 0.001 and 0.004). Mwü p values in the same range were also obtained between Groups 6 and each of Groups 1, 2 and 4 in the C_2 score, and again between Group 6 and some of these groups in the C_3 , MIN_{34} , MIN_{13} as well as in all the MAX scores. Only in the D scores other than D_{12} did no significant differences appear (p values between 0.264 and 0.888).

The close proximity of Group 6 to Group 3 in the MAX_{12} score together with the large number of strong contrasts just cited makes it possible to assert that the difficulties created by the verbal representation were comparable to those of the symbolic representation. The verbal representation, it is true, relied on simple and meaningful sentences as stimulus material. At the same time it presupposed some knowledge of grammatical rules, which became incorporated into the total rule system of the representation through the adjunction of conditional rules for effecting the correct transformations of

the sentences. The conditional rules were what Game B and Game D had in common¹. But this highly abstract connection was clearly of little use to subjects whose performance in Game B was very low both before and during the transfer test².

No. 5 - Changes in both the type of representation and in the material.

Game G was substituted for Game C in the transfer test of Treatment Group 8. Game G used the iconic representation of Game C with the machine of cycle 2 replaced³ by a machine of cycle 3. The present comparison is therefore concerned with contrasts between Group 8 and Groups 1, 2, 3 and 4.

The ranks of the mean residual scores⁴ of Group 8 show a good deal more variation than those of Groups 5, 6 or 7. Group 8 led all or most of the other four groups in the C_2 , C_4 , MIN_{1234} , MAX_{12} , D_{34} , D_{1234} scores and trailed behind in the C_3 , MIN_{34} , MIN_{13} , MAX_{1234} , MAX_{34} , MAX_{13} and D_{13} scores. In the MAX_{12} score it contrasted favourably with Group 3 (Mwu $p = 0.000$) and was as good as Group 1, the best of the four groups using Game C (Mwu $p = 0.973$), so that its knowledge of Game B was in no way deficient. In the C_3 and MIN_{13} scores, on the other hand, the Group 8 residuals ranked below those of Group 6. The strongest contrasts in disfavour of Group 8 appeared in the C_3 score, where comparisons with Groups 1, 3 and 4 yielded MwU p values of 0.000, 0.000 and 0.004 respectively. Low p values reinforcing this contrast

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- 1 Chapter 3, p.102. The conditional rules for Game D were slightly simpler than those for Game B.
 - 2 Inspection of the raw mean scores in Appendix S5 - 3, p. shows that Groups 3 and 6 had the lowest IL and C_1 scores. The discrepancy in their C_3 scores endorses the contrast between Game C and Game D.
 - 3 Chapter 4, p.121.
 - 4 Appendix S5 - 1, p.287.

were also observed by comparison with Group 3 in the MIN_{13} and MAX_{34} scores (0.002 and 0.001 respectively), and Group 1 in the MAX_{13} score (0.004). Against this p values in the C_2 and C_4 scores did not reach lower than 0.35.

A possible explanation for the peculiar features noticeable in these contrasts is that a change in the cycle of one of the machines created an additional complication which depressed performance in the third column, but that this complication counterbalanced the difficulties normally encountered in the second and fourth columns¹, where performance depends on translation. For Group 8 translation no longer involved matching of the individual steps in each task as these steps were no longer in one: one correspondence. It is apparent from the raw mean scores² that in the case of Group 8 the difference between the C_3 score and either of the C_2 or C_4 scores was far less marked than in the case of Groups 1 to 6, of whom step-by-step translation was demanded.

No. 6 - Changes in the type of representation without changes in the stimulus material.

Treatment Group 7 used Game F as the new representation in the transfer test. Game F differed from Game B only in that the machine of cycle 2 was replaced by a machine or cycle 3. The same numbers served as stimulus material in both games.

Both the mean residual scores³ and the p values obtained from the individual comparisons leave little doubt that Group 7 stood favourably apart from all the other groups, both in the overall level of its performance and in the ease with which

1 Cf. p.162 supra.

2 Appendix S5 - 3, p. 298.

3 Appendix S5 - 1, p. 287.

adaptation to the new representation was made. The number of strong contrasts recorded¹ in its favour at p values of less than 0.005 was 55. Although the mean residuals of the MAX₁₃, MAX₁₂₃₄, MAX₁₂ and C₃ scores throw some doubt on the unchallenged superiority of Group 7, only two contrasts in them were observed at a p level below 0.100, and of these the lowest was 0.018 in favour of Group 1 in the MAX₁₃ score. The four scores whose position is less exalted than that of the rest tell us that, given the high raw scores² of Group 7 in the three covariates of the regression, certain features in the total performance were less outstanding than others. But in all the MIN and D scores, as well as in the C₂ and C₄ scores, the superior position of the group was unaffected by the regression.

From the methodological point of view Group 8 stands between Group 7 and the four groups using the iconic representation of Game C. The present comparison revealed more contrasts³ at p values below 0.005 between Group 7 and Group 8 than between Group 8 and any one of the other four groups. This suggests that a change in the stimulus material of the unfamiliar representation is the dominant factor in determining the ease or difficulty of transfer. Such an interpretation would be consistent with what has already been observed in the other comparisons. In Comparison No. 2 the changes in the machine reminder charts had little effect, but in Comparison Nos. 3 and 4 the replacement of the iconic representation by the symbolic and verbal representations for

1 Appendix S5 - 2, p. 290.

2 Appendix S5 - 3, p. 298.

3 Appendix S5 - 2, p. 290.

Groups 5 and 6 respectively depressed performance considerably. In the case of Group 6, moreover, similarities in the rules of Games B and D did little to offset the change in stimulus material. In the case of Group 7 no change in the stimulus material was introduced and the rule systems of the two representations overlapped through the retention of one of the machines in both games. No noticeable difficulties resulted and the two representations seem virtually to have been treated as one¹.

Four hypotheses arising out of the discussion of Section C are summarised in Table 5 - 15.

1 From the mathematical point of view it would be possible to regard the representation of Game F as an extension of the representation of Game B. In that event the new machine of cycle 3 would be a redundant generator, but the problem of learning the new rules of this generator would still remain.

Table 5 - 15

The Hypotheses suggested by the Results of Section C.

Content	Evidence
C1. In the age range 7 - 12 there is no appreciable difference in transfer performance between the 3 junior and the 3 senior years.	Comparison No.1.
C2. The degree to which relations between two representations are made explicit in a transfer test has no appreciable effect on performance.	Comparison No.2.
C3. A change in the stimulus material of the unfamiliar representation is the dominant factor in determining the ease or difficulty of transfer.	Comparison Nos.2 to 6 together.
C4. Stress during transfer is least if the same stimulus material is retained for both representations, and the difference between them arises only from differences in their rule systems and their types.	Comparison Nos.2 to 6 together.
C5. If a change in the stimulus material and the rule system is such that the translation of tasks involves their step-by-step matching,	Comparison Nos. 3 and 4.
(continued)	

Content	Evidence
then transfer to an iconic representation that displays the 4 vertices of the tetrahedron is much easier than either to a symbolic or to a verbal representation. C6. If a change in the stimulus material is accompanied by a change in the rule system that precludes the possibility of step-by-step matching of tasks during translation, the performance level in these tasks rises, but between tasks that are initiated independently in either type of transfer this same level falls.	Comparison No.5

Hypothesis C1 suggests that transfer tasks create a challenge throughout the full age range of the secondary school and that the prevalent concentration on the primary range is unjustifiably narrow. Many striking differences in transfer performance arising from changes in the second representation actually occurred in the upper age range between 10 and 12. The results contained in Hypothesis C2 seem contrary to expectation, given that the tests were wholly unfamiliar. At the same time they are encouraging

as evidence that pupils were thinking their way through each task rather than relying on superficial perceptual cues for the right answers when such cues were made available. Nevertheless, thinking must be initiated by the stimuli of the representation that are perceived. Hypotheses C3 to C6 show how this initiation depended on the nature of the second representation. Hypotheses C3 and C4 stress the importance of the stimulus material in the reception of information. If it is left intact while the rule system and type¹ of the representation are changed, no interference seems to arise and the stress during transfer is extremely small. Hypotheses C5 is the only one whose outcome might have been predicted, namely that with the type of the representation held constant transfer is easiest² when the second representation can be associated with the geometric properties of the tetrahedron in a natural way. From Hypothesis C6 it appears that the effects of a change in the type of the representation offset the difficulties arising from step-by-step translation. These difficulties were very noticeable in the analysis of Section A and will be further investigated in the Section which follows.

If we write S for a change in the stimulus material, R for a change in the rule system, T for a change in the type of the representation, and also place bars over these letters when no change has occurred, then an eight-fold

1 Chapter 3, pp. 101 - 102.

2 Inspection of the raw mean scores in Appendix S5 - 3 on p.298 shows that Treatment Group No.3 had the highest mean gain ($MAX_{13} - C_1$) relative to the familiar representation of Game B.

classification of transfer is created. In this classification, however, the combination $\bar{R} \bar{T}$ cannot be realised in practice, since it is impossible to change the type of the representation without changing its rule system at the same time. Hence $S \bar{R} \bar{T}$ and $\bar{S} \bar{R} \bar{T}$ cannot be experimentally investigated. Again, $\bar{S} \bar{R} \bar{T}$ stands for transfer between a representation and itself and is therefore trivial. Of the remaining 5 categories in the classification $S \bar{R} \bar{T}$ was realised in the transfer tests of Groups 1 to 6, $\bar{S} \bar{R} \bar{T}$ in the transfer test of Group 7, and $S \bar{R} \bar{T}$ in the transfer test of Group 8. $S \bar{R} \bar{T}$ could have been realised through the mere substitution¹ of one set of stimuli for another, possibly of 4 different shapes in place of the present ones in the frames of Game C. The somewhat unusual $\bar{S} \bar{R} \bar{T}$, on the other hand, could have been realised if the two axes of rotation of the tetrahedron had been changed from axes in space² to axes in the body. The shapes in the frames of Game C would then have been manipulated according to a different set of rules. The lack of information arising from these omissions weakens the generality of Hypothesis C3. Some compensation for the omission of $S \bar{R} \bar{T}$ is provided by Comparison No. 4, where the rule systems of Game B and Game D closely resembled one another. As far as the omission of $\bar{S} \bar{R} \bar{T}$ is concerned, changing the axes to axes in the body would again, in general, have precluded subjects from matching up the individual steps in each task. It therefore seems reasonable to suppose that the results for $\bar{S} \bar{R} \bar{T}$ would have been similar to those for $S \bar{R} \bar{T}$, as provided by Group 7.

1 Chapter 1, p. 54.

2 Chapter 3, p. 93. See also Appendix S3 - 3, p.227.

Section D.

Contributions of the Test Items to the Score Characteristics.

The Equivalence Matrix in which the scores were recorded consisted of 5 rows and 4 columns¹. In what follows these rows will be labelled downwards from A to E², and the columns in the usual manner from 1 to 4. The transfer tests between representations³ of Type (2,3) produced entries in the matrix as follows: The tasks were one-step tasks for Row A and two-step tasks for Row B. Every task consisted of finding the final state given the initial state and the machine(s), regardless of whether it was a leading task or the translation of a leading task. The leading tasks produced entries in Columns 1 and 3, while the translated tasks produced entries in Columns 2 and 4. The entries in Rows C, D and E were produced by tasks consisting of 2, 3 and 4 steps respectively. In these tasks, however, there was a change in pattern between the leading and the translated tasks. In the leading tasks the starting and the final states were given, and subjects had to discover the most economical sequence of machines by which they were connected. In the translated tasks, however, they exchanged the machines for the corresponding machines in the other representation and then applied the new machines to a given starting state to find an unknown final state.

In the transfer tests between Games of different Type these tasks could not be constructed on the same pattern⁴.

1 Chapter 4, p. 126.

2 The letter C will therefore always refer to a row, whereas in the previous sections it was used as a general abbreviation for 'column'.

3 See the task analysis of the transfer test between Game B and Game C in Appendix S4 - 4, p. 262.

4 Appendix S4 - 7, p. 280. See also Chapter 4, p. 119.

For this reason the scores of Treatment Groups 7 and 8 were excluded from the comparisons of the present Section, and are only referred to in passing when the occasion demands. The number of available subjects dropped from 460 to 361 as a result.

Three comparisons were made between the following selections of Rows, in such a way that the possible maximum for each pair of compared scores was always the same:

1. Row B with Row C.
2. Rows A and B together (AB) with Row D.
3. Rows B and C together (BC) with Row E.

To provide an overview and common source of reference the mean scores of all the scores obtained have been listed in Table 5 - 16 at this stage. Each score is expressed as a raw score and also as a percentage of the maximum for the Row in which it occurs.

Table 5 - 16

The Mean Scores of the Individual Entries in the Equivalence Matrix.

Rows	Columns				Maximum for each entry
	1	2	3	4	
A	0.94 (94%)	0.90 (90%)	0.95 (95%)	0.94 (94%)	1
B	1.86 (93%)	1.67 (84%)	1.66 (83%)	1.67 (84%)	2
C	1.86 (93%)	1.19 (60%)	1.60 (80%)	1.13 (57%)	2
D	2.42 (81%)	1.44 (48%)	2.36 (79%)	1.72 (57%)	3
E	2.27 (57%)	1.52 (38%)	2.32 (58%)	1.60 (40%)	4

It is apparent from this Table that the loss in performance arising from translation was far greater in Row C than in Row B. Highly significant¹ differences in favour of Row B were observed between the B2 and C2 scores, and again between the B4 and C4 scores (Wcx. $p = 0.000$ in both cases). The B1 and C1 scores were virtually equal (Wcx. $p = 0.896$), and the difference between the B3 and C3 scores was also far from being significant (Wcx. $p = 0.296$). As measures of the partial stresses the absolute values of the differences (B1 - B2), (B3 - B4) and (B1 - B3) and of the corresponding differences for Row C were computed. Pairwise comparisons between each type of difference then confirmed that translation caused far more stress in Row C (Wcx. $p = 0.000$ for translation into the new representation, and 0.020 for translation from the new representation). On the other hand, the stresses arising from transfer between independent tasks were statistically equal (Wcx. $p = 0.829$). Since the B1 and C1 scores were so close together and since the tasks producing the C2 and C4 scores were designed on the same pattern as all the tasks in Row B, the loss during translation must be ascribed to differences² in the thought processes that were needed to complete the leading and translated tasks of Row C.

-
- 1 In this Section, as in Sections A and B, the number and the p values of the individual comparisons that were judged significant at the usual level of 0.05 were small enough to prevent the risk of a Type I error rising above this level.
 - 2 An independent examination of the corresponding results obtained from Treatment Groups 7 and 8 showed no appreciable discrepancies between initiated and translated tasks in the first two columns of Rows B and C, and actually some gain in favour of the translated task in the fourth column of Row B.

The picture is elaborated by the frequency distributions of Table 5 - 17. A substantial majority of subjects lost nothing between columns 1 and 2, 3 and 4, or 1 and 3. The inequalities show that far more subjects lost during translation in Row C than in Row B, and that in both rows fewer subjects gained than lost when starting a comparable task in the new representation. As many as 31 subjects in Row B gained when they translated back into the familiar representation, but in Row C this phenomenon occurred¹ only once.

Table 5 - 17

The Frequency Distribution of the Order Relations
between Pairs of Scores in Row B and Row C.

No. Ss		No. Ss		No. Ss	
40	B1 > B2	7	B1 < B2	314	B1 = B2
122	C1 > C2	0	C1 < C2	239	C1 = C2
30	B3 > B4	31	B3 < B4	300	B3 = B4
89	C3 > C4	1	C3 < C4	271	C3 = C4
48	B1 > B3	13	B1 < B3	300	B1 = B3
58	C1 > C3	11	C1 < C3	292	C1 = C3

¹ It could only occur if subjects correctly found the machine sequence before translation, recorded it incorrectly on paper and then produced the correct translation.

No. 2 - Rows A and B together (AB) with Row D.

As might be anticipated from Table 5 - 16 differences between the (AB) scores and the D scores were highly significant in each Column (Wcx. $p = 0.000$ in all cases). The two three-step tasks in Row D that did not depend on translation were more difficult than their counterparts in Rows (AB) because of the more complex strategy involved in the search for the sequence of three machines¹. Equally significant in favour of Rows (AB) were also differences in the stresses experienced during the translation of these tasks.

The stresses between the independent tasks of Columns 1 and 3, however, favoured Row D (Wcx. $p = 0.007$). Inspection of the percentages of the mean scores also shows a marked drop between Column 1 and Column 3 in Rows B and C, but not in Rows D and E. It would seem, therefore, that with an increase in task difficulty and in practice performance in columns 1 and 3 tended to equality, although the overall level of performance declined. Subjects adjusted to the new representation more easily than to the more difficult tasks.

The frequency distributions of Table 5 - 18 again show equality to be the dominant relation. Among the frequencies of the inequalities the highest one was 138 in Row D between columns 1 and 2, where subjects translated a difficult task into the unfamiliar representation. For translation back into the familiar representation the frequency dropped to 88, but both these large numbers go to confirm that translation

1 An appreciable drop in the mean percentage was also observed in the combined scores of Treatment Groups 7 and 8, but only in Row E, presumably because Group 7 contained many high achievers.

in Row D caused more stress than in Rows (AB) combined. In Row D as in Row C hardly any subjects gained as the result of translation, but 52 subjects gained and as many lost points between Columns 1 and 3. This corroborates what was said about the tendency of subjects to adjust to the new representation as they passed through the test.

Table 5 - 18

The Frequency Distribution of the Order Relations
between Pairs of Scores in Rows (AB) and Row D.

No.Ss		No.Ss		No.Ss	
59	(AB)1 > (AB)2	17	(AB)1 < (AB)2	285	(AB)1 = (AB)2
138	D1 > D2	1	D1 < D2	222	D1 = D2
36	(AB)3 > (AB)4	38	(AB)3 < (AB)4	287	(AB)3 = (AB)4
88	D3 > D4	1	D3 < D4	272	D3 = D4
58	(AB)1 > (AB)3	24	(AB)1 < (AB)3	279	(AB)1 = (AB)3
52	D1 > D3	52	D1 < D3	257	D1 = D3

No. 3 - Rows B and C together (BC) with Row E.

As far as the individual Columns are concerned the combination (BC) was strongly favoured by all tests of significance (Wcx. $p = 0.000$ throughout). The longer tasks in Row E produced much less positive response. The stresses between the Columns, however, present a very different picture.

Table 5 - 19

The Frequency Distribution of the Order Relations
between Pairs of Scores in Rows (BC) and Row E.

No.Ss		No.Ss		No.Ss	
140 90	(BC)1 > (BC)2 E1 > E2	2 1	(BC)1 < (BC)2 E1 < E2	219 270	(BC)1 = (BC)2 E1 = E2
97 91	(BC)3 > (BC)4 E3 > E4	25 0	(BC)3 < (BC)4 E3 < E4	239 270	(BC)3 = (BC)4 E3 = E4
90 64	(BC)1 > (BC)3 E1 > E3	19 61	(BC)1 < (BC)3 E1 < E3	252 236	(BC)1 = (BC)3 E1 = E3

In Table 5 - 16 the difference between the highest and the lowest mean percentage score is 36 for Row C alone, whereas for Row E it is only 15. In Row E also the percentage drop arising from translation is much lower than in Rows C or D, and between the independent tasks of Columns 1 and 3 of Row E the percentage actually rises 1 point in favour of Column 3 after dropping in all other rows except Row A, and quite steeply in Rows B and C. By the time they reached Row E subjects were adjusting better both to translation and to transfer between independent but comparable tasks in each representation. Although Row E contained the long four-step tasks the stresses arising from translation were no greater than for Rows (BC) combined (Wcx. $p = 0.282$ for Columns 1 and 2, and 0.830 for Columns 3 and 4). Between the independent tasks of Columns 1 and 3 the stress in Row E was

decidedly lower (Wcx. $p = 0.001$). Table 5 - 19 also shows that in Row E subjects gained in column 3 over column 1 more than anywhere else.

Wilcoxon tests comparing the stresses between Columns 1 and 2 with those between Columns 3 and 4 revealed significant differences only in the case of Row C ($p = 0.004$) and Row D ($p = 0.000$), the stress arising from translation into the new representation being the greater one in both cases. In the case of Row B differences between the two directions of translation are suggested by Table 5 - 16. Statistically, however, these differences fell short of significance (Wcx. $p = 0.174$), because according to Table 5 - 17 a very large number of subjects experienced no stress from translation whatsoever.

Section D has localised some of the information contained in the Hypotheses of Section A to specific items and to 6 out of the 8 Treatment Groups. Two additional Hypotheses arising from the discussion are stated in Table 5 - 20.

The Hypotheses suggested by the Results of Section D

<u>Content</u>	<u>Evidence</u>
D1. When the completion of a task before translation involves a strategy of selection and the subsequent translation does not, the loss in performance during translation is greater than if this strategy is absent.	Highly significant differences in Comparison No. 1 between the Row B and Row C in the stresses arising from translation.
D2. If tasks demanding use of the above strategy are arranged in order of increasing difficulty the level of performance drops with difficulty, but performance tends to become more equally distributed over the Equivalence Matrix as subjects pass through the test. This tendency begins with a decrease in stress between the independent tasks and culminates in a decrease in stress between tasks and their translations, as well as in an equalisation of this stress between the two directions of translation.	Comparison Nos. 2 and 3. together.

Hypothesis D1 pinpoints the conditions under which translation was most difficult. On the one hand translation under these conditions merely involves the execution of routine operations with the mates¹ of the machines whose sequence has already been found in the initial task, on the other hand it calls for a radical re-arrangement of the thought processes² between the completion of these tasks. In Row B of Table 5 - 16 no such arrangement was required, so that the percentage drop between the first two Columns must there be ascribed to the unfamiliar representation as such. There is no corresponding drop between any of the other Columns.

Hypothesis D2 states that adjustment to this translation did occur despite the progressive increase in task difficulty, but that it occurred more slowly than adjustment to the new representation in the independent tasks. Whether in a longer series of transfer activities complete equalisation of performance over the four columns would occur before or after tasks had been mastered at every level of difficulty remains a matter for further investigation. Hypothesis D2 merely suggests that at a certain stage task difficulty begins to dominate difficulties arising from transfer. Within a longer series it might be possible to decide whether more progress could be made towards perfect psychological equivalence if an experimental group of pupils were allowed to complete each leading task in the representation of their choice as soon as equality of performance in Columns 1 and

-
- 1 The correspondence between the machines in the two representations was given in every transfer test.
 - 2 Evidence that pupils were thinking rather than looking for perceptual cues was also provided in the discussion of Hypothesis C2 on p. 172.

3 had been demonstrated, and also to abstain from translation as soon as the same equality appeared. A saving in time would result and this would allow a greater range of more difficult tasks to be attempted. Towards the end of the session the full use of the Equivalence Matrix would be re-introduced to see whether more transfer or more practice with a greater variety of tasks was productive of a higher level of equally distributed performance.

In conclusion it should be noted that Row A in Table 5 - 16 provides none of the differential information obtainable from the other rows. It was the only row containing one-step tasks. The uniformly high performance in these tasks serves as a control to the effect that differences appearing in subsequent rows did not arise from a misunderstanding of any of the basic principles on which the transfer tests were constructed.

Section E.

The Determination of a Distance Measure

A measure $d(X,Y)$ which could appropriately be called the distance¹ between two representations, X and Y , would have to satisfy the following three axioms² :

1. $d(X,Y) = 0$ if $X = Y$, and $d(X,Y) > 0$ if $X \neq Y$,
2. $d(X,Y) = d(Y,X)$,
3. $d(X,Y) + d(Y,Z) \geq d(X,Z)$, where Z is any third representation.

The D_{1234} measure of total stress and unequally distributed performance can not serve as a measure of distance because the differences of which it is composed are not necessarily differences arising from different representations. If every row of the Equivalence Matrix had its highest score in column 1 and its lowest score in column 4 then all these differences would be differences between a leading and a trailing task in the same representation.

If only columns 1 and 3 were retained and the translation of tasks were ignored in a transfer test then the first two axioms would be satisfied by the D_{13} score. The statement $X = Y$ in axiom 1 would then be interpreted as meaning that ignorance and knowledge were equally distributed over the two representations. Also, since all the D scores are

-
- 1 Dienes (1963), p. 69, makes brief reference to the possible construction of such a measure without, however, providing any details. In the present construction the underlying space must be thought of as a psychological space since the values of the distance measure are entirely dependent on the reaction of subjects to the two representations, not on any properties of the representations as such.
 - 2 See, for example, Gregson (1975), p. 106, where metric spaces are discussed in connection with judgments of similarity. If axiom 3 does not hold then $d(X,Y)$ is termed a semimetric.

computed as differences between MAX and MIN scores, they can never assume negative values.

If the same or comparable¹ tasks appeared in a third representation Z, axiom 3 would also be satisfied. Each set of these scores obtained from corresponding tasks would be represented by three non-negative integers on a segment of the number line, for which the absolute values of the differences between pairs of numbers provide a Euclidean metric. The D_{13} scores for the whole test would then be obtained by summing each of the three terms in the inequality of the axiom over the tasks, and by this summation the inequality would be preserved.

When the whole Equivalence Matrix is introduced used and translation also is taken into account the measure $d(X,Y)$ defined as the sum $(D_{12} + D_{13} + D_{34})$ will still satisfy axioms 1 and 2. If $d(X,Y) = 0$, then the equal distribution of performance extends to all four columns. The symmetry of axiom 2 is guaranteed because the two stresses resulting from translation in each direction have been incorporated into the sum, so that one and the same test would provide measures of $d(X,Y)$ and $d(Y,X)$ equally well².

This time, however, the $d(X,Y)$, $d(Y,Z)$ and $d(X,Z)$ scores obtained from transfer between three representations

-
- 1 When translated tasks also are present the tasks in columns 1 and 3 can only be comparable, not identical, since the translated task in column 2 is already identical with the task in column 1. In the absence of translation the untranslated tasks in each representation can be made identical. This change would strengthen the validity of the D_{13} measure but in no way affect the validity of axiom 3.
 - 2 The test must start with translation in one of the directions. But since the directions alternate throughout the test, the resulting discrepancy is not a serious one.

need not satisfy axiom 3 on the strength of their definition. To see this we have only to suppose that the D_{12} and D_{34} scores are zero in two out of the three d scores, and that they increase the third d score¹ to the point where it exceeds the sum of the other two. The truth or falsity of axiom 3 is therefore a matter for experimental determination. It is with this determination that the present Section is concerned.

To ensure that subjects could serve as their own controls and that task comparability would approach task equality as closely as possible three forms of a Special Transfer Test were designed from the representations of Games B, C and D, as shown in Table 5 - 21. Successive pairs of tasks starting with Task No. 1 were identical in content in every form. Tasks Nos. 1 to 6 were two-step tasks without decision points², whereas Task Nos. 7 to 12 were three-step tasks with decision points whenever the representation was leading. In this way each task could appear in each of the three representations as a leading task and also twice as a trailing task, with each of the other two representations leading in turn. For the sake of simplicity the leading task was repeated before the second translation and subjects were instructed merely to

1 Performance in a translated task affects only one of the D_{12} or D_{34} scores. Performance in an independent task affects one of these scores and also the D_{13} score. If transfer is made between three representations taken in pairs, performance in an independent task will affect two of the D_{13} scores and also one of the D_{12} or D_{34} scores in each of the two translations which proceed from it.

2 Chapter 4, p. 108.

copy out what they had written the first time. The three forms were randomly distributed to 112 subjects in Grades 7 and 8 in such a way that each form was used about the same number of times in each class. Each subject worked through the Initial Learning Test before taking the Special Transfer Test. Violations of axiom 3 occurred as shown in Table 5 - 22.

Table 5 - 21

The Three Forms of the Special Transfer Test between Games B, C and D.

Task No.	FORM I Leading/Trailing		FORM II Leading/Trailing		FORM III Leading/Trailing	
1.	C	B	B	C	D	C
2.	C	D	B	D	D	B
3.	B	C	D	C	C	B
4.	B	D	D	B	C	D
5.	D	B	C	B	B	C
6.	D	C	C	D	B	D
7.	C	D	B	D	D	B
8.	C	B	B	C	D	C
9.	B	D	D	B	C	D
10.	B	C	D	C	C	B
11.	D	B	C	D	B	D
12.	D	C	C	B	B	C

Table 5 - 22

The Number of Subjects whose Scores did not conform to Axiom 3.

	No. of subjects
$d(B,C) + d(C,D) < d(B,D)$	9
$d(B,C) + d(B,D) < d(C,D)$	25
$d(B,D) + d(C,D) < d(B,C)$	7

Since only one of these inequalities can apply to the scores of a given subject the violations of axiom 3 occurred in the case of 41 out of 112 subjects. Most frequently it was violated by the high¹ d score between Games C and D, and because of this bias it was also violated by the mean d scores of these 41 subjects. The mean scores of the total group of 112 subjects, on the other hand, conformed to axiom 3 and may therefore be represented by the triangle of

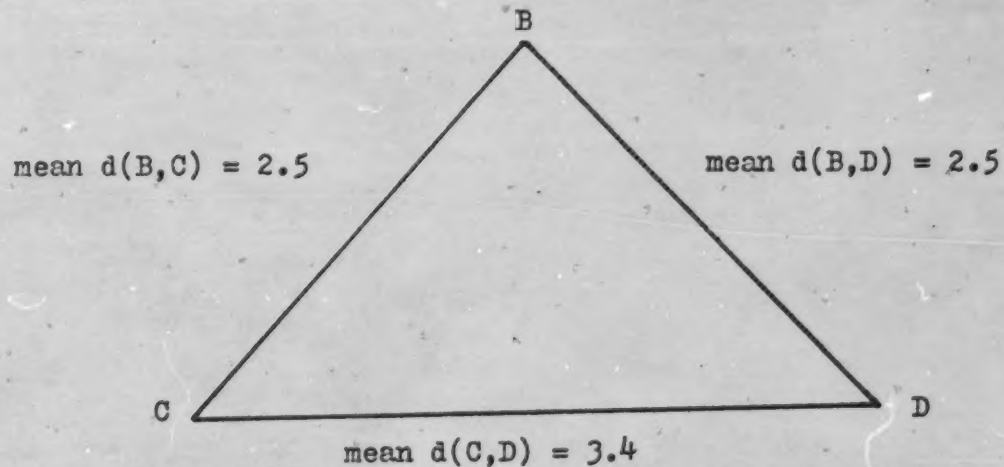


Figure 5 - 1

Isosceles Triangle illustrating the applicability of Axiom 3 to the mean distance scores.

¹ Cf. the different standing of these two Games in the comparisons of Section C.

Fig. 5 - 1. The results suggest that for a sufficiently large number of subjects the means of the d scores will have all the properties of a metric¹ whereas the d scores of an individual may only have the properties of a semi-metric. (Hypothesis E1).

In Figure 5 - 1 the distances between 3 representations form a triangle in 2 dimensional space. If a fourth representation were added we could expect to obtain a tetrahedron in 3 dimensional space, and in the general case of n representations a simplex in $(n - 1)$ dimensional space. It might, however, happen that after a certain number of representations had been reached there was no further increase in the number of dimensions required to accommodate the simplexes as the number of their vertices increased. The appearance of this constraint would then provide information about the number of dimensions to be ascribed to the psychological space in which the distances had been generated². As long as an n point simplex requires $(n - 1)$ dimensions we only know that the number of dimensions of the psychological space is equal to or greater than $(n - 1)$.

-
- 1 The metric $(D_{12} + D_{13} + D_{34})$ considered here could be generalised to a metric of the form $(D_{12}^n + D_{13}^n + D_{13}^n)^{1/n}$ for different positive integer values of n . To what extent, if any, this would make axiom 3 applicable to the scores of more individuals remains an open question.
 - 2 See Coombs (1964), p. 143. Since the d measure here has arisen from pairs of representations, it constitutes, in his terminology, a proximity between points of the same set.

CHAPTER 6

CONCLUSIONS AND POSSIBILITIES

The analyses of the last Chapter suggest that the tests engaged the interests and efforts of all the participants. The procedures were understood in principle and at the same time involved the completion of a series of graded tasks whose complexity was sufficient to produce a wide range of responses. Conclusions were drawn from a substantial number of hypotheses whose directions could, for the most part, be statistically well established.

The use of the Equivalence Matrix,¹ the numerical observations that may be obtained from this use, and the types of hypotheses that admit of formulation in consequence are all clearly applicable to many contexts in mathematics, languages or perception.¹ But any claims to generality that are advanced for the directions of the hypotheses and the conclusion based on them must be viewed against the limitations imposed on the experiments by the context in which they were conducted. As far as could be ascertained no precedent for the use of the Equivalence Matrix or the formulation of the present hypotheses seems to exist, and the latter for the most part only suggested themselves during the analysis of the results obtained. Under these conditions the conclusions elicited from the directions of the hypotheses are at best tentative, or to use Tukey's term,² 'indications' designed to stimulate and direct research in

1 Chapter 1, pp 55 - 59.

2 Tukey (1962), p.46.

the future. Concepts are clarified and frameworks for investigations established, whose adoption or modification may yield more strongly supported results at a later stage, when the size and variety of the experimental field have become appropriately enlarged.

The present experiments ranged over 6 consecutive age groups taken from the upper part of the school population, and involved the use of 6 representations. The presence of this much diversity can support some claim to generality for the hypotheses advanced in Sections A, B and D, especially as the effects of differences in age were discounted in Hypothesis C1¹. Again, Hypothesis C5² is supported by what is already known about pupils' responses to iconic representations from many other contexts.

However, two obvious limitations suggest themselves. In the first place it is not possible to extrapolate to age levels below those actually tested, because the potential for performance in younger children undergoes radical changes with their cognitive development. Secondly the experiments centred around a single mathematical structure, whose tasks and representations could all be mastered within the span of two classroom periods. There is no reason to suppose that in contexts where such mastery is dependent on much longer periods of practice the validity of all the present conclusions would go unchallenged.

The large number of variables uncovered can give rise to

1 Chapter 5, p. 170.

2 Ibid., p. 170.

a correspondingly large number of questions, to which satisfactory answers could only be provided by systematic and extensive research. Such research might be motivated by the desire to provide a coherent account, with support from existing psychological theory, of the dependence of psychological equivalence and all its associated variables on the selected mathematical structure, on the stimulus material and the rules of its representation, and finally on the large number of subject variables which are commonly taken into consideration in the study of learning. Or again it might be motivated by the desire to find some immediately relevant and fruitful applications of some or all of the present procedures to the classroom. These motivations are overlapping and complementary. But they are also distinct, in that not every result obtained in the interests of theory has immediate applications, and in that applications often prove useful in practice long before a justification in some theoretical framework can be found for them.¹ The present experiments were conducted in an endeavour to create a methodology for comparing the understanding of different representations. The potential for investigation as offered by the Equivalence Matrix was exposed to the fullest possible extent, without regard for theoretical or practical predilections. This neutrality notwithstanding some of the approaches and results appear of more immediate relevance to the classroom than others. The special status of translation which emerged from the analyses of Sections A and D suggests that an obvious simplification could be introduced

¹ The distinction here closely parallels the distinction between theory and application in medical science.

by the teacher if he confined activities to transfer between independent tasks in each representation. Theoretically this would amount to teaching or testing only a part of psychological equivalence, but at a given stage the omission of the other part might well serve his purpose best. Again, in the absence of directives to the contrary from research, one might expect teachers to have little inclination to use the elaborate scoring system in its entirety or to concern themselves with the distance measure of Section E in any serious way.

The ability to pass between representations and to understand how they relate to one another is part of the accomplishment of the professional mathematician. Besides this, pedagogy has long recognised the value of resorting to alternative modes of explanation when understanding has failed, or of challenging this understanding by demanding an alternative approach to an already correct solution. The issue can therefore never centre on the presence of psychological equivalence as a component in the learning of mathematics, but only on the extent to which this component should be singled out for special attention at different stages of the curriculum, and on the form that this attention should take. When viewed superficially the use of multiple representations would seem to amount to a slowing down of the rate of exposition and learning, since every item of mathematical content has to be manipulated at least twice. To justify the expenditure of time and materials resulting from the explicit attention to psychological equivalence research

must somehow demonstrate that short and preferably long term gains will be produced by the investment. Some justification has been provided by the literature, but only for a limited range of the school population and the curriculum. The format of activities suggested by the Equivalence Matrix would bring the conditions of transfer between representations under strict control, and within the confines of the present experiments the value ¹ of positive response to these conditions was to some extent established.

It is thus fair to say that some evidence already exists to support the view that transfer between representations should not simply be trusted to occur when it is needed, but that it can promote understanding if it is set before pupils as an objective through directed activities specially designed for the purpose. At the same time this evidence has not settled in any conclusive manner questions concerning the nature and number of representations that should be used, concerning the period of time over which their usage should extend, or concerning the type of pupil who would benefit most as a result. If this kind of transfer is accepted as a valuable activity in principle, then its acceptance could still be liberal or conservative in form. Liberal acceptance ² would presumably introduce all pupils to such transfer in the interest of their conceptual development and treat the use of multiple representations as an integral part of the curriculum. The risk of this policy is that unnecessary duplication of activities may be demanded from pupils who

1 Hypothesis B6, Chapter 5, p.157.

2 Liberal acceptance has consistently been recommended by Dienes. See, for example, Dienes (1960), p.44.

have already abstracted well and are ready to pass on to a new range of mathematical concepts. Conservative acceptance, on the other hand, would presumably confine the use of such transfer to pupils who had revealed serious misunderstanding of some part of mathematics in the conventional representation. This would amount to a switch over from prevention to cure and only involve ad hoc modifications to the curriculum for those who could not pass through it as it stood. The risk of waste would be removed, but replaced by a risk of a different kind, namely that wrong attitudes might be established before misunderstanding of the conventional representation was detected by the teacher,¹ and that correct abstraction through successful transfer to a new representation would become correspondingly more difficult.

Regardless of whether the liberal or conservative policy were adopted it would become necessary to obtain more detailed knowledge than exists today about the ways in which pupils react to different pairs of representations in juxtaposition. Although there may well be several representations of the same structure which elicit the same reaction in juxtaposition, it is evident from the present results² that some juxtapositions create far more stress than others.

1 If the criterion for understanding in just the conventional representation consists of nothing more than the ability to perform routine manipulations that very closely resemble those that have been taught, then some pupils may well satisfy this criterion. Their misunderstanding may only be detected much later, when they are required to use the information they are supposed to have gained in another context.

2 Hypotheses C3 to C6, Chapter 5, pp 170 - 171.

Juxtapositions which result in very little or no stress seem the least likely ones to promote understanding. On the other hand, juxtapositions that are productive of an excessive amount of stress may prove equally ineffective because they confront the pupil with too large a psychological distance¹ to be traversed. The problem thus consists in finding juxtapositions that are productive of an optimum amount of stress, one that will promote learning and reduce as the learner progresses..² It could happen that the two representations whose juxtaposition created excessive stress were both conventional ones such as the decimal and fractional representations of rational number.³ Neither representation could now be discarded. Instead attempts could be made to see whether the stress between them could be reduced if a sequence of suitably chosen didactic representations were placed between them as a bridge and gradually withdrawn.

The need for more detailed knowledge about pupils' reactions to juxtapositions creates the core of enquiry for the theorist. His task would seem to be a fourfold one, the

1 Cf. Dienes (1963), p.93.

2 Cf. Hypothesis D2, Chapter 5, p.182. If learning of the tetrahedral group had continued beyond the present experiments, it might have been more economical to continue with practice of the difficult 4 step tasks in just one representation, or to allow each pupil to complete each task in the representation of his choice. Possibly the insistence on the use of multiple representations and translation is most helpful in creating the initial understanding of the elements and operations of the structure. Once this initial understanding has been gained a different type of stress is created by the task complexity. This may be present irrespective of the representation that is employed, and under these circumstances the continued use of multiple representations would seem to have little purpose.

3 Chapter 1, p.39.

construction of a classificatory scheme for representations, the systematic exposure of how differences in the categories created by this scheme make transfer between representations easy or difficult, the investigation of the growing capacity for the matching of representations as part of human cognitive development, and finally the appraisal of the value of transfer in mathematics education.

Two categories for the classification of representations were employed in the present Thesis, the stimulus material and the rule system by which this material becomes organised.¹ In view of the criticism² that may be levelled at the three-fold classification of representations into concrete, iconic and symbolic, it might be more satisfactory to replace this classification by a classification of the stimulus material into multi-sensory or uni-sensory on the one hand, and into suggestive or non-suggestive on the other, according as the content is or is not reflected in its physical properties. The classification of the rule system is a more intricate matter. For the representations in use here conditional rules were distinguished from unconditional rules.³ This distinction could provide one category of classification, but obviously the number,⁴ generality and complexity of the rules

1 Chapter 1, pp 49 - 53.

2 Ibid., p. 29.

3 In Games B, D and F the rules were conditional.

4 The number of rules may, but need not, vary with the size of the structure. Much will depend on the rules that are selected and on the generality of their formulation. For instance, the cyclic group of order 3 and the cyclic group of order 12 could be represented by 3 and 12 shapes in frames respectively, together with a single rule stating that a machine moves these shapes one place to the right on each application unless they are on the extreme right, in which case they are moved to the first place on the left.

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must also be taken into account. Although the complexity of mathematical representations is less than that of languages, a satisfactory development here could well require the attention of linguistics.

Once satisfactory categories for the classification of representations had been established the second task would be to analyse the effects on transfer created by differences in the individual categories, and also by different combinations of these differences. Systematic experimentation here is impeded by the fact that for a given structure and stimulus material the rule system cannot be varied arbitrarily,¹ that most of the differences arising within the classification are not scalable but only nominal,² and that there is no method independent of subjects' reactions by which these differences might be combined so as to yield a single difference that could be quantified.³ In consequence levels cannot be assigned to the variables in advance of experimentation and no economy can be effected in this experimentation through interpolation between such levels. Any pattern of conclusions which might ultimately be found to remain invariant

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- 1 In the case of the tetrahedral group it was possible to find the rule systems of Game B and Game F for the 12 numbers they had as common stimulus material.
 - 2 The stimulus material could be scaled on size but not on shape. Variations in size between representations are not of interest in general. The rules could be scaled on their number, but this may well be the least important of their properties.
 - 3 The question of how subjects combine different dimensions of the stimulus material has been fully discussed by Garner (1973), pp 97 - 121, in the area of perception, where nothing corresponding to the rule system of a representation is present. In Section E of Chapter 5 this difference was quantified on the strength of the subjects' reactions.

over classes of representations ¹ has therefore to be established from the results of a series of suitably varied experiments.

As far as the theorist's third task is concerned, psychological equivalence as a measurable phenomenon is easily defined as equality of performance ranging over a sample of content in two representations. In contrast any analysis of the cognitive processes which make this performance possible must be difficult, in view of the amount of co-ordination that these processes entail. They must be more complex than the processes responsible for performance in a single representation, because the performer has now also to take the relationships between the two representations into account. From the static point of view he has to recognise ² identity against a background of all irrelevant differences in the stimulus material and rules of the two representations. From the dynamic point of view he must conceive, before execution, certain transformations of the stimulus material from one representation into the other, in conformity with a double system of rules, and in such a way that the identity of content is preserved.

Conceptual development and emerging representational competence ³ have been extensively studied from the genetic

1 It could be expected that invariance would also be confined to classes of subjects, possibly narrower ones than the age group participating here. This, however, would place no undue restrictions on the applicability of the results to the classroom.

2 Cf. Hayes & Simon (1977), p.40.

3 Chapter 1, p.24 - 27.

point of view along Piagetian lines. The approach can be extended to psychological equivalence,¹ but possibly with the proviso that controlled experimentation here is preferable to observation. Whereas the young child will spontaneously reveal knowledge of concepts and their representations to the observer, he is less likely to attempt any matching of representations under the same conditions. In a controlled experimental situation, on the other hand, his efforts become analysable in detail by the protocol,² in which stepwise progress towards the completion of a task may be systematically recorded by the experimenter. The protocol then contains information on how the task was completed or left uncompleted in each representation, so that divergence in the approach to the goal no less than in its attainment may be allowed to contribute to psychological inequivalence. To date the emphasis in the use of the protocol seems to have been placed on the attainment, in each representation, of a structural goal such as the total ordering of a number of elements³ from the stepwise accumulation of knowledge about their partial orders through questions from the subject and answers from the experimenter. If this emphasis were in some way shifted to the matching of the representations whenever a goal or sub-goal had been reached in one of them, much detailed information could be gained about the ways in which subjects effected their transformations of the stimulus material and about the causes which prevented them from doing so. One factor that has suggested itself as the

1 See, for example, Gillieron (1977), *passim*.

2 Gillieron (1977), p. 109.

3 Ibid., pp. 189 - 190.

result of this type of analysis is the subject's expectation¹ regarding the suitability of the stimulus material to serve in the particular representation. There seems to be no reason why through answers to a suitably constructed system of questions all important factors² involving both subject variables and features of particular representations, should not ultimately be revealed.

The completion of the first three tasks should leave the theorist better equipped for offering advice in the classroom, where the goals and approaches required for their attainment are as a rule more complex than in the psychological laboratory. With many of the variables identified he should be in a position to address himself not only to the study of psychological equivalence in mathematics as it is in itself, but also to the ways in which this equivalence relates to the objectives that are conventionally regarded as terminal³ in mathematics education.

If progress from the simple to the complex serves the interests of both methodology and economic expenditure, then he could commence his investigations by searching for a vindication of the conservative policy in the first instance.

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- 1 Ibid., p. 201. The sentences of Game D in the present experiments might justly have been regarded by subjects as unsuitable for the representation of a mathematical structure.
 - 2 The approach through the simplex suggested at the end of Section E, Chapter 5, could at best only provide an upper bound to the number of factors involved. Their identification would still remain a matter for experimentation.
 - 3 Whereas in the learning of foreign languages psychological equivalence is an end in itself, it is never more than the means to an end in either the learning or practice of mathematics.

Where there is failure in understanding the remedial value of a new approach is easily assessed. Pupils in need of remedial work could be introduced to transfer between representations on an individual basis. Under these conditions the protocol and all other resources of the psychological laboratory could be made available to them. Once sufficient knowledge about the remedial effects of transfer had been gained, the activities that had led to success could be committed to print. In this way a large number of pupils requiring the same kind of attention would be reached by the output of a small group of experimenters who had chosen to specialise in the formation of mathematical concepts and skills through transfer.

After that might follow a study of the distribution of transfer ability in the school population at large. The object of this study would be to ascertain the extent to which this ability was related to mathematical attainment at all levels. and whether tests measuring this ability were measuring components of mathematical ability which already available tests left untapped. This would be a more costly undertaking and involve the construction of transfer tests resembling the present ones in format. The structures selected¹ should be remote from the regular curricula, to reduce the biases created by different curricula and their interpretations in different schools.

1 Group or logic structure suggest themselves. In the selection of these structures and still more in the selection of their representations an arbitrary element would be bound to be present. It is, after all, also present in the design of every mathematics curriculum. But if there is no obvious superficial relationship between the contents of the tests and the contents of the various curricula, then the assumption that the tests would retain their validity even if the curricula were changed gains considerably in strength.

SPECIAL APPENDICES

TO

CHAPTER 3

Success with the construction of some standardised tests measuring psychological equivalence through transfer would be a recommendation for passing on to the liberal policy, which seeks to improve mathematical attainment as measured by these tests through regular practice.¹ To be vindicated in its turn this policy would ultimately have to produce a higher level of performance by a sizeable experimental population in the public examinations.

It is through the projection of future developments along these lines that the pursuit of psychological equivalence might be pleaded. Clearly this pursuit would become more effective in proportion as it could draw on the co-ordinated expertise of mathematicians, psychologists, teachers and educational administrators from all over the world. Through co-operation of this kind many of the difficulties that now surround the subject could be removed and a clearer view of the function of multiple representations in mathematics learning obtained. Another step towards improving the quality of instruction would then have been taken. The present Thesis was written in the belief that the serious interest already shown in this direction over the past twenty years would continue to grow, and that this growth would be promoted if some procedures with the potential for generating a substantial amount of information were to hand.

1 This practice would range over the structures found in the curricula. Both conventional and didactic representations could be employed.

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A P P E N D I C E S

SPECIAL APPENDICES

TO

CHAPTER 2

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SPECIAL APPENDIX S2 - 1

Approximate Correspondences between Grades and Age

Levels in North American Schools.

AGE	GRADE
5 - 6	Primary (Kindergarten)
6 - 7	1
7 - 8	2
8 - 9	3
9 - 10	4
10 - 11	5
11 - 12	6
12 - 13	7
13 - 14	8
14 - 15	9
15 - 16	10
16 - 17	11
17 - 18	12

SPECIAL APPENDIX S2 - 2

The Cuisenaire Apparatus

This consists of number-pieces from one to ten units in length. The pieces are unsegmented, and their colour-coding has a special significance. The unit rod is white and the seven rod is black, but the other rods are grouped into families of related numbers: the reds (vermillion - two units, crimson - four units, brown - eight units), the blues (light green - three units, dark green - six units, blue - nine units), and the yellows (yellow - five units, orange - ten units). In the initial stages great stress is laid on the use of the colour relations to form correct equations such as $y(\text{yellow}) + r(\text{red}) = b(\text{black})$, without explicit reference to the numerical relations as given by the lengths. Further details will be found in Cuisenaire & Gattegno (1954), also in E.S.A. (1976).

SPECIAL APPENDIX S2 - 3

The Stern Apparatus

The basic apparatus consists of a set of number-pieces from one to ten units in length. The pieces are coloured to facilitate discrimination, and also segmented so that component units can be easily recognised. Insight into cardinal number is thus gained through measuring rather than through the counting of discrete objects.

Further details and a description of the various accessories will be found in Stern (1971).

See also E.S.A. (1976).

SPECIAL APPENDIX S2 - 4

The Dienes Algebraic Experience Materials

These consist of (i) a balance beam with weights to be placed in slots at various distances from the fulcrum, (ii) a peg board with coloured pegs, and (iii) a collection of flat pieces of wood cut out in the shapes of squares, rectangles, equilateral triangles, trapezia and rhombuses. Appropriate arrangements of the materials can be used to teach a large variety of number properties ranging from the commutative law of multiplication to the solution of linear equations and the factorisation of perfect squares.

For further details see E.S.A. (1976).

GAME A

The algebraic stimulus material used here is the same as would be used in standard textbooks. The algebraic symbols in their proper combinations represent the essentials of the structure of the tetrahedral group. If we denote the identity transformation by I then the Machine Description tells us that $X^3 = I$, $Y^2 = I$, and $YX^2Y = YXY$. These three equations are sufficient¹ to define the entire group.²

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1. These or an equivalent set of three equations are also necessary.
 2. From these equations and the two Machine Reminder charts which list their consequences the group table on page 95 may be constructed as follows: Let $T_1 =$ the identity transformation (i.e. X^3 or Y^2), $T_2 = X$, $T_3 = Y$, $T_4 = X^2$, $T_5 = YX$, $T_6 = XYX^2$, $T_7 = YX^2$, $T_8 = XY$, $T_9 = X^2YX$, $T_{10} = YXY$, $T_{11} = YXY$ and $T_{12} = X^2Y$. To find, for example, the single transformation which is equivalent to T_6 followed by T_5 we imagine that T_6 has been applied to the frame marked START and taken us to the frame marked XYX^2 on either chart. To this frame we now apply the transformation T_5 which has been defined as equal to the Y transformation followed by the X transformation. We therefore locate the frame XYX^2 on the Machine Reminder for the Y Machine and find that this Machine transforms this frame into the frame X^2YX . Next we locate the frame X^2YX on the Machine Reminder

continued.../

They constitute a concise representation stripped of all inessentials. Most of the properties of the structure are left implicit. For the purpose of short-term experimentation it was therefore thought advisable to unfold these properties on the Machine Reminder charts.

The three equations tell us that any stimulus material that is endowed with a rule system in which transformations exactly corresponding to the X and Y transformations can be described will form a representation of the tetrahedral group, irrespective of the ways in which these transformations are described in detail. The X and Y transformations thus set a general pattern to which every representation of type (3,2) must conform. This pattern is easily detected in the representations of Game B, Game C and Game D.

...continued/

for the X Machine and find that this Machine transforms this frame into the frame YXY. The two transformations are therefore equivalent to the transformation corresponding to YXY. In the above list this transformation is the transformation T_{11} . This transformation now appears inside the table in the square where the row commencing with T_6 on the left meets the column headed by T_5 . All other entries may be found in a similar manner, it being understood that if the START frame is reached by a sequence of transformations then this sequence is equivalent to T_1 , the identity transformation.

MACHINE DESCRIPTION FOR GAME A

In this Game there are two MACHINES, the X MACHINE and the Y MACHINE. The letters in each frame tell you which machines have been applied to the START frame to get you to the frame in the smallest number of moves.

The X MACHINE works like this: If you put a frame into the X MACHINE then the letters in the frame are multiplied by X on the right. If you multiply by X on the right three times then you return to the frame with which you started.

The Y MACHINE works like this: If you put a frame into the Y MACHINE then the letters in the frame are multiplied by Y on the right. If you multiply by Y on the right two times then you return to the frame with which you started.

When you apply either of these two machines to the frames you sometimes reach a frame which could have been reached from the START frame by a different route in a smaller number of moves. When this is the case the shorter sequence of letters is shown in the frame.

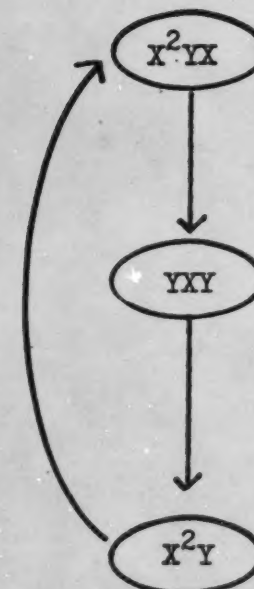
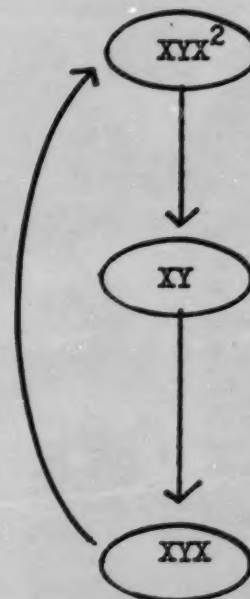
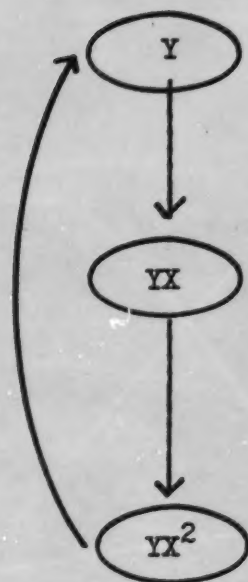
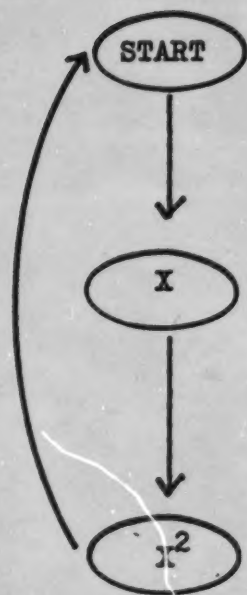
For example, if you put the frame YX^2 into the Y MACHINE you would expect to get the frame YX^2Y . Instead you find that the frame you get is labelled YXY .

So you have the equation $YX^2Y = YXY$. This equation is used to shorten the letter sequence wherever it is possible.

If you look at your MACHINE REMINDER sheets you will see how groups of letters are changed into one another by the two MACHINES.

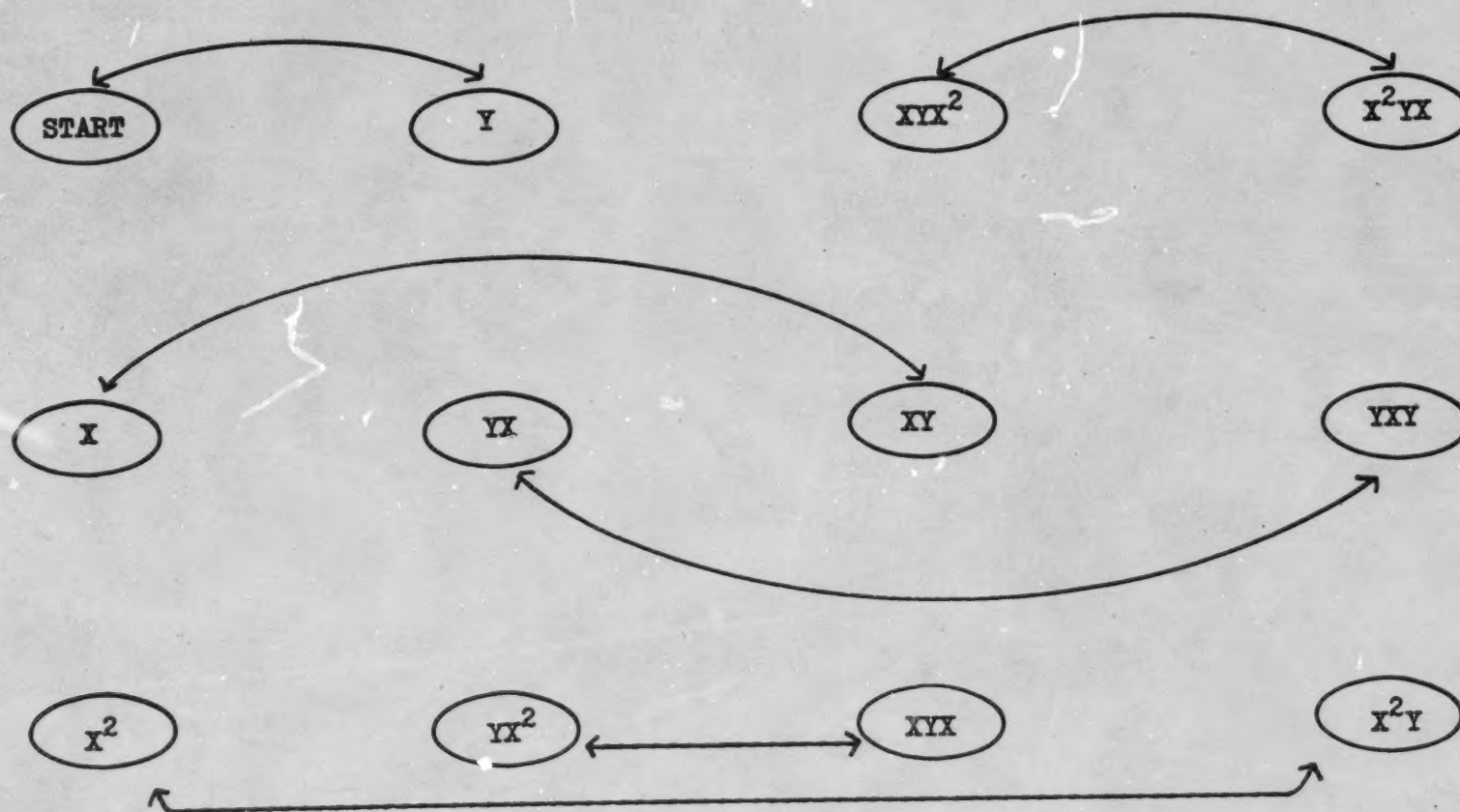
MACHINE REMINDER FOR GAME A

The X MACHINE moves the letters around like this



MACHINE REMINDER FOR GAME A

The Y MACHINE moves the letters around like this



GAME B

In this Game the transformations are based on the familiar number operation of multiplication. The prime and composite factors of 60 together make up 12 code numbers for the positions of the tetrahedron and each transformation is expressed in terms of one or more multiplications. To create transformations in exact conformity with the pattern of Game A the multiplications performed by the two Machines are made to depend¹ on the properties of the individual numbers instead of being the same for all of them. Each Machine is thus a conditional function.² This feature adds complexity to the description of the representation but not to the two charts, in which the transformations resulting from the description are all laid out.

1 This representation was suggested by Dr. Z.P.Dienes in a private communication.

2 A simple example of a conditional function occurring in school mathematics is the absolute value function $y = |x|$. This function is equal to $y = x$ provided x is greater than or equal to zero, and equal to $y = -x$ provided x is less than zero.

MACHINE DESCRIPTION FOR GAME B

In this Game there are two MACHINES, the TWO-FOUR MACHINE and the SWITCH MACHINE.

The TWO-FOUR MACHINE multiplies by 2 all those factors of 60 which can not be divided by 4. The other factors it divides by 4.

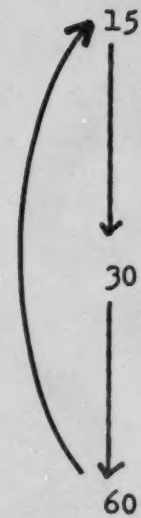
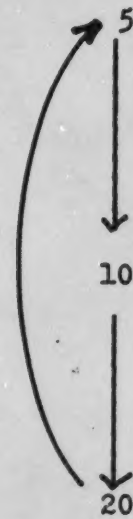
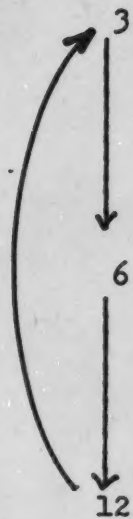
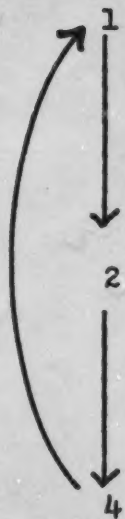
The SWITCH MACHINE works as follows :

- (1) Every odd factor of 60 it multiplies or divides by 3, whichever is possible.
- (2) Every even factor of 60 which does not divide by 4 it multiplies or divides by 5, whichever is possible.
- (3) Every even factor of 60 which divides by 4 it multiplies or divides by 3 and also by 5, whichever is possible.

If you look at your MACHINE REMINDER sheets you will see how the factors of 60 are changed into one another by the two MACHINES.

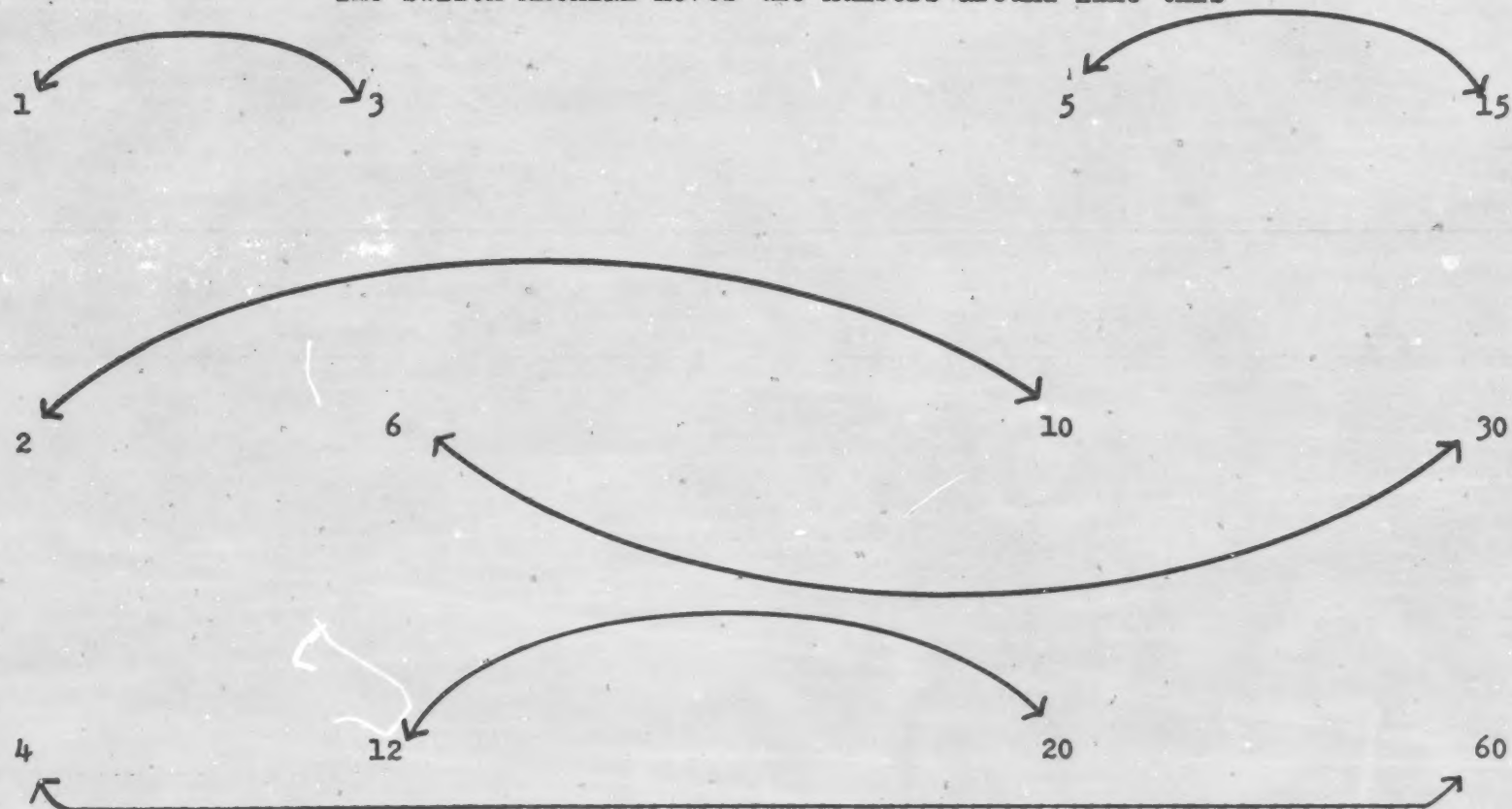
MACHINE REMINDER FOR GAME B

The TWO-FOUR MACHINE moves the numbers around like this



MACHINE REMINDER FOR GAME B

The SWITCH MACHINE moves the numbers around like this



SPECIAL APPENDIX S3 - 3

GAME C

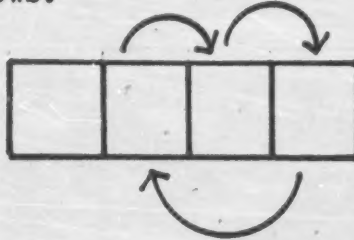
In this Game the fixed frame represents the box into which the tetrahedron can be fitted in 12 ways, and the four shapes inside the frame ¹ are the code labels for the 4 vertices. The description of the Machines is straightforward and the two charts are wholly iconic in character. No prior knowledge of mathematics is required for following through the sequence of operations on the shapes, which again conforms exactly to the pattern of Game A. It will be noticed that the description of the Machines relates to the positions of the shapes in the frame, not to the shapes as such. This means that the axes about which the tetrahedron is rotated are here regarded as fixed in space. ²

-
- 1 This representation was used, without the charts, in a comparative study of several group structures in a research project conducted by Dr. Z.P. Dienes at the University of Sherbrooke, Quebec, under Canada Council Grant R - 17 - 77 - 400 - 5 - F in 1974.
- 2 If the axes were moving with the body then the Machines would have to be described in relation to the shapes inside the frame. For example, the Machine of order three might then be called the Circle Machine, and the Switch Machine of order two might always exchange the circle with the square and the triangle with the hexagon, regardless of their positions in the frame.

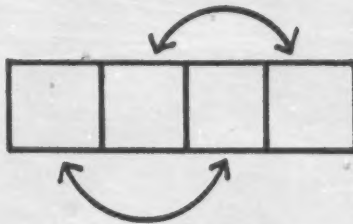
MACHINE DESCRIPTION FOR GAME C

In this Game there are two MACHINES, the FIRST MACHINE and the SWITCH MACHINE.

The FIRST MACHINE leaves the shape in the first position where it is and moves the other shapes around as is shown here by the arrows:



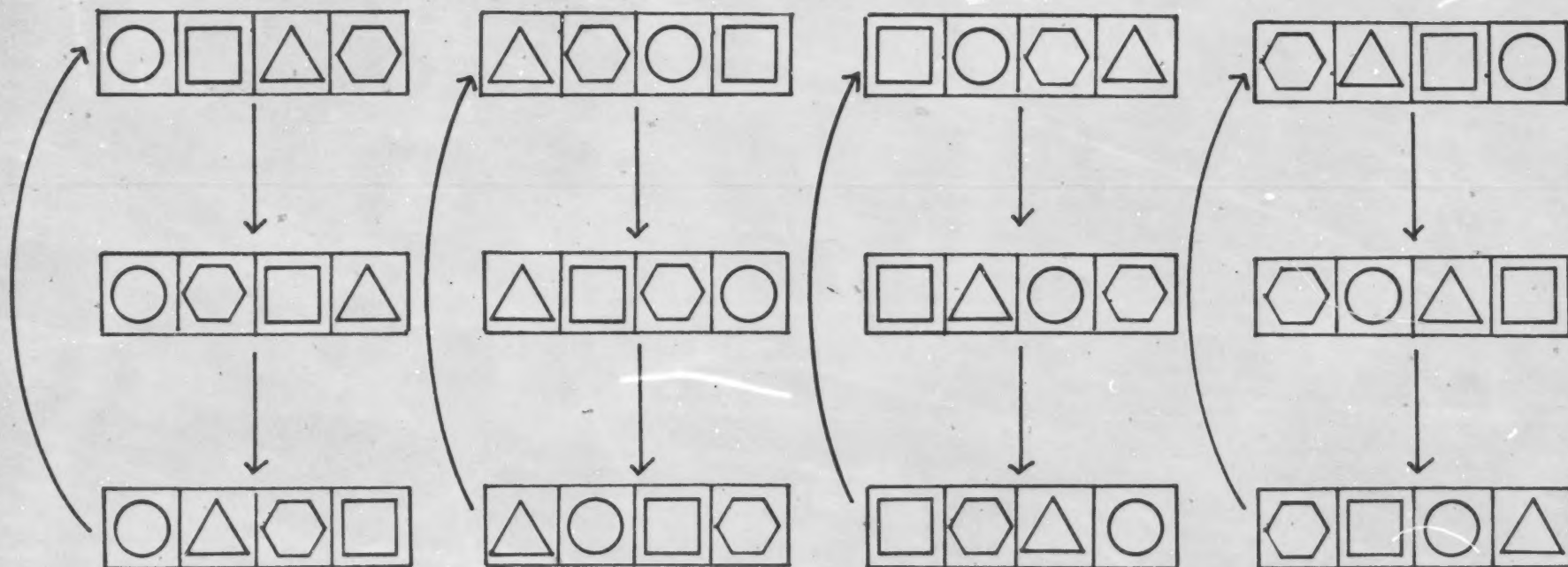
The SWITCH MACHINE switches the shapes around in pairs as is shown here by the arrows:



If you look at your MACHINE REMINDER sheets you will see how the frames are changed into one another by the two MACHINES.

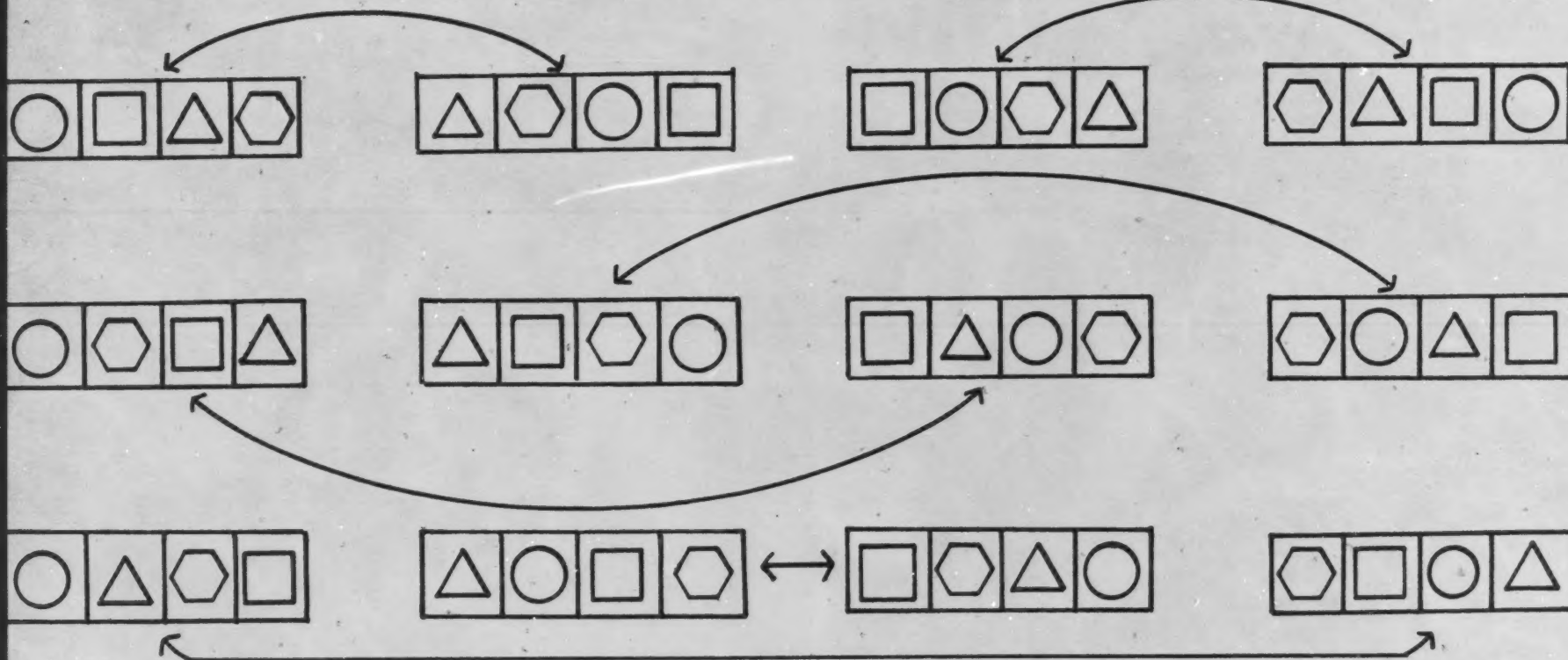
MACHINE REMINDER FOR GAME C

The FIRST MACHINE moves the shapes around like this



MACHINE REMINDER FOR GAME C

The SWITCH MACHINE moves the shapes around like this



SPECIAL APPENDIX S3 - 4

GAME D

This Game presupposes some elementary knowledge of grammar. The Machine of order three passes the verb through three tenses, and the Machine of order two changes the meaning of the sentence. Both Machines are again conditional functions. Just as in Game B the use of such functions is an artifice through which a wide range of stimulus material can be adapted to serve as a representation of the group.

MACHINE DESCRIPTION FOR GAME D

In this Game there are two MACHINES, the TIME MACHINE and the SWITCH MACHINE.

The TIME MACHINE changes the time (tense) of the verb from present to future, from future to past, and from past to present.

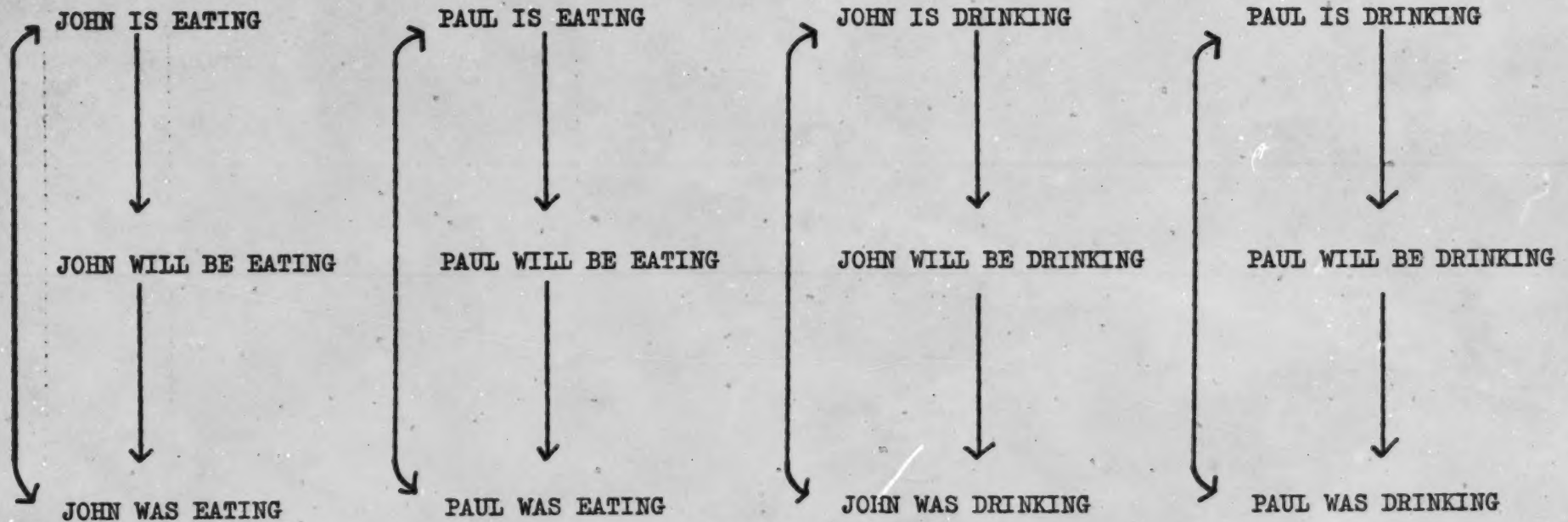
The SWITCH MACHINE works as follows:

- (1) If the verb is in the present this machine changes the boy's name.
- (2) If the verb is in the future this machine changes the verb's meaning.
- (3) If the verb is in the past this machine changes the boy's name and also the verb's meaning.

If you look at your MACHINE REMINDER sheets you will see how the sentences are changed into one another by the two MACHINES.

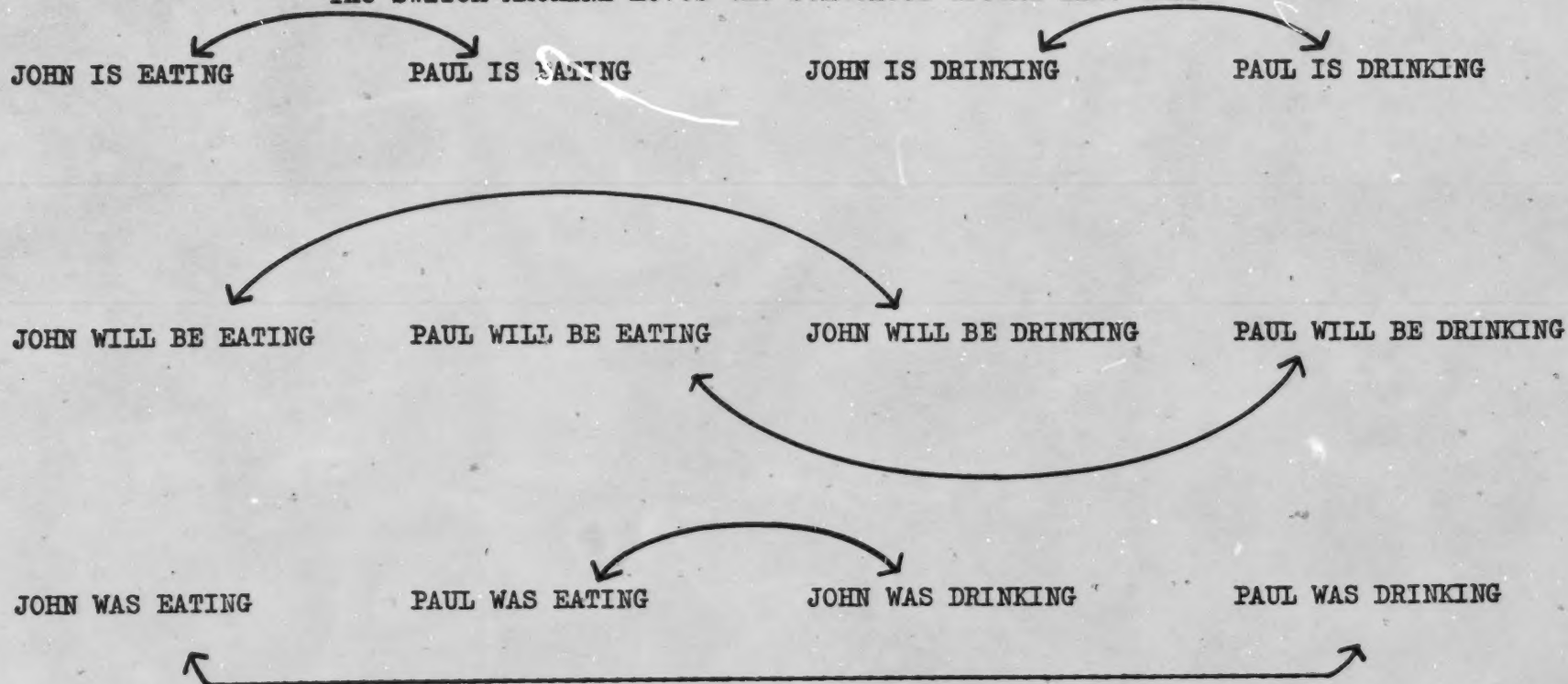
MACHINE REMINDER FOR GAME D

The TIME MACHINE moves the sentences around like this



MACHINE REMINDER FOR GAME D

The SWITCH MACHINE moves the sentences around like this



SPECIAL APPENDIX S3 - 5

GAME E

This Game differs from Game A in that the Y Machine of order two has been replaced by the Z Machine of order three. The X Machine is the same in both Games. Mathematically this change amounts to a change of one of the generators. An inspection of the charts for Game A reveals that the transformation XY is of order three. Consequently we can identify XY with Z and relate the two Games by the equation $XY = Z$.¹ On multiplying both sides of this equation by X^2 on the left we obtain $X^3Y = X^2Z$, and since X^3 is the identity transformation this reduces to $Y = X^2Z$. Y is thus expressed in terms of X and Z. Since both Games completely describe the same group the table of the 12 transformations on page 95 could equally well be constructed from the present Game.

If we denote the identity transformation by I then the Machine Description tells us that $X^3 = I$, $Z^3 = I$, and $XXZ = Z^2$. These three equations² are again sufficient to define the entire group and set the pattern to which any representation of type (3,3) must conform. This pattern is detectable in Game F, Game G, and Game H.

-
- 1 This identification is in keeping with the construction of all the other Games of each type. It is most easily seen from a comparison of the charts for Game C with the charts of Game G.
 - 2 The third equation by which X and Z are related could be replaced by $ZXZ = X^2$. The given equation transforms into the last equation if both sides of it are multiplied by Z on the left and then by X^2 on the right.

MACHINE DESCRIPTION FOR GAME E

In this Game there are two MACHINES, the X MACHINE and the Z MACHINE. The letters in each frame tell you which machines have been applied to the START frame to get you to the frame in the smallest number of moves.

The X MACHINE works like this: If you put a frame into the X MACHINE then the letters in the frame are multiplied by X on the right. If you multiply by X on the right three times then you return to the frame with which you started.

The Z MACHINE works like this: If you put a frame into the Z MACHINE then the letters in the frame are multiplied by Z on the right. If you multiply by Z on the right three times then you return to the frame with which you started.

When you apply either of these two machines to the frames you sometimes reach a frame which you could have reached from the START frame by a different route in a smaller number of moves. When this is the case the shorter sequence of letters is shown in the frame.

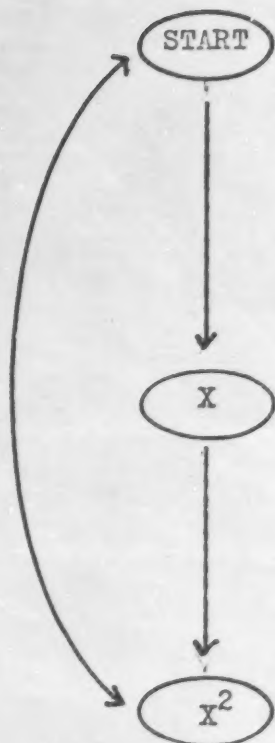
For example, if you put the frame XZ into the Z MACHINE you would expect to get the frame XZX. Instead you find that the frame you get is labelled Z².

So you have the equation $XZX = Z^2$. This equation is used to shorten the letter sequence wherever it is possible.

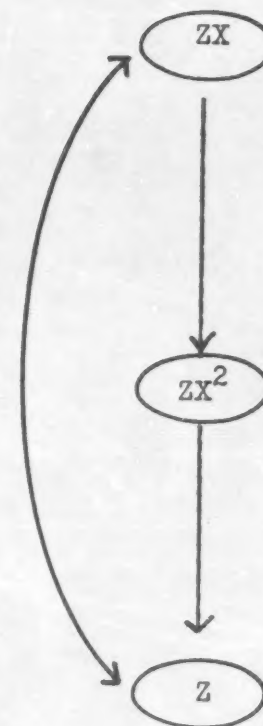
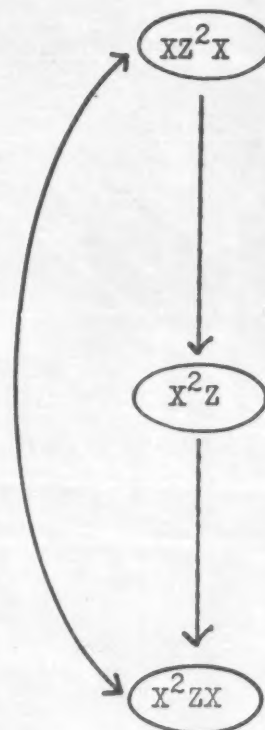
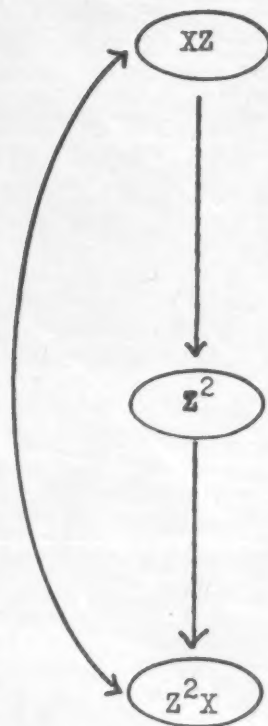
If you look at your MACHINE REMINDER sheets you will see how the groups of letters are changed into one another by the two MACHINES.

MACHINE REMINDER FOR GAME E

The X MACHINE moves the letters around like this



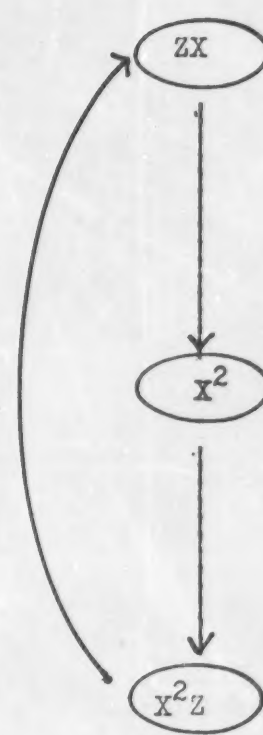
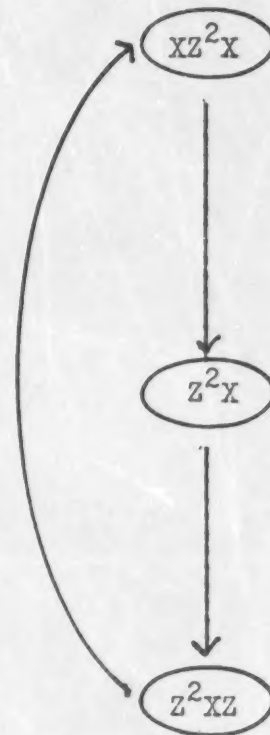
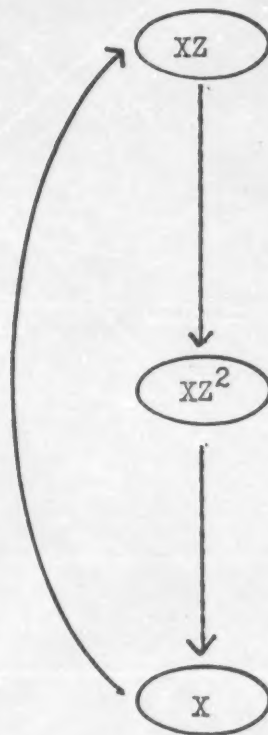
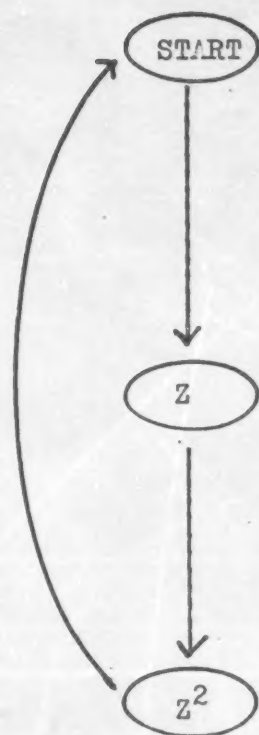
263



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MACHINE REMINDER FOR GAME E

The Z MACHINE moves the letters around like this



SPECIAL APPENDIX S3 - 6

GAME F, GAME G and GAME H

These three Games are of type (3,3) and use the stimulus materials of Game B, Game C and Game D respectively. In each Game one of the Machines from the type (3,2) Game with the same stimulus material is retained. The other Machine is exchanged for a Machine of order three. In Game F and Game H the new Machines are again conditional functions. In Game G the Machine of order three that suggested itself in a natural way was the Second Machine. This works like the First Machine, except that it keeps the shape in the second position where it is and permutes the other three shapes cyclically in the frame.

The names of the First Machine (Games C and G), the Second Machine (Game G), the Switch Machine (Games B, C and D), the Time Machine (Games D and H), and the Two-Four Machine (Games B and F) give some indication as to the changes which these Machines are designed to effect. No correspondingly suitable name suggested itself for the Machines which replaced the Switch Machine of Game B and Game D in Game F and Game H respectively. Both Machines were therefore named Special Machines.

MACHINE DESCRIPTION FOR GAME F

In this Game there are two MACHINES, the TWO-FOUR MACHINE and the SPECIAL MACHINE.

The TWO-FOUR MACHINE multiplies by 2 all those factors of 60 which can not be divided by 4. The other factors it divides by 4.

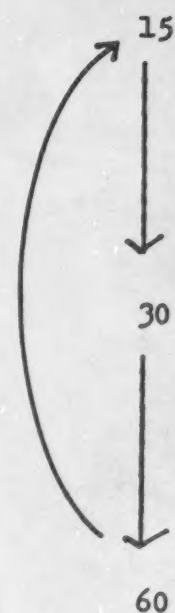
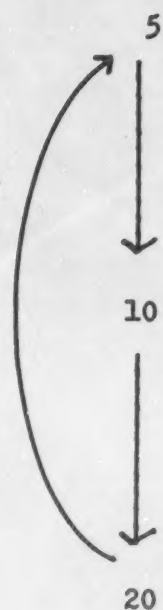
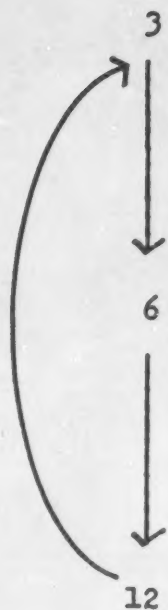
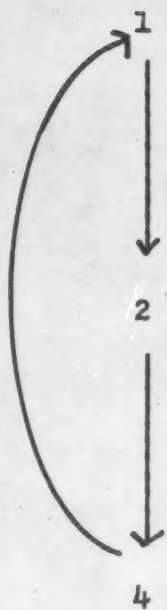
The SPECIAL MACHINE works as follows:

- (1) Every odd factor of 60 it multiplies by 4, divides or multiplies by 3, and divides or multiplies by 5, whichever is possible.
- (2) Every even factor of 60 which does not divide by 4 it divides by 2, and divides or multiplies by 3, whichever is possible.
- (3) Every odd factor of 60 which divides by 4 it divides by 2, and divides or multiplies by 5, whichever is possible.

If you look at your MACHINE REMINDER sheets you will see how the factors of 60 are changed into one another by the two MACHINES.

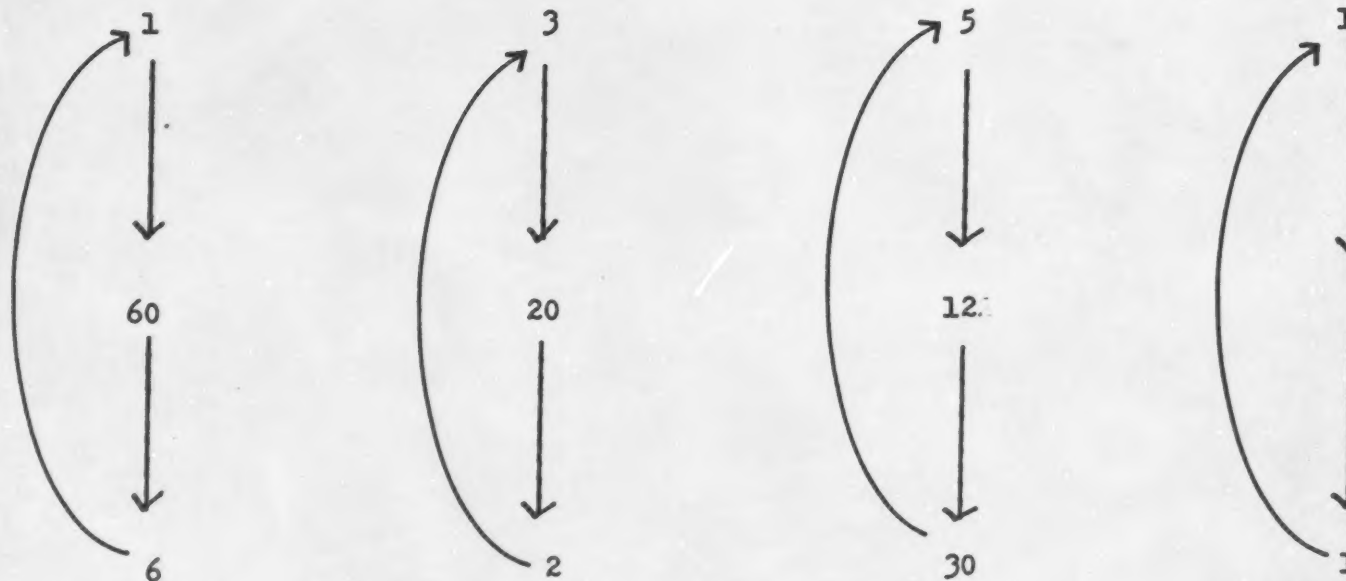
MACHINE REMINDER FOR GAME F

The TWO-FOUR MACHINE moves the numbers around like this



MACHINE REMINDER FOR GAME F

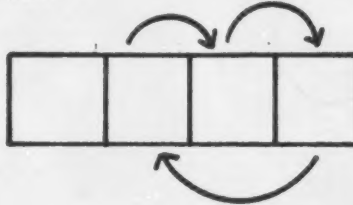
The SPECIAL MACHINE moves the numbers around like this



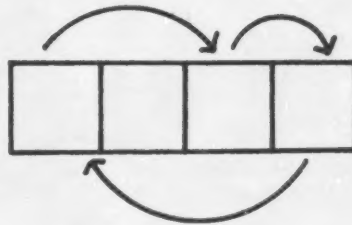
MACHINE DESCRIPTION FOR GAME G

In this Game there are two MACHINES, the FIRST MACHINE and the SECOND MACHINE.

The FIRST MACHINE leaves the shape in the first position where it is and moves the other shapes around as is shown here by the arrows:



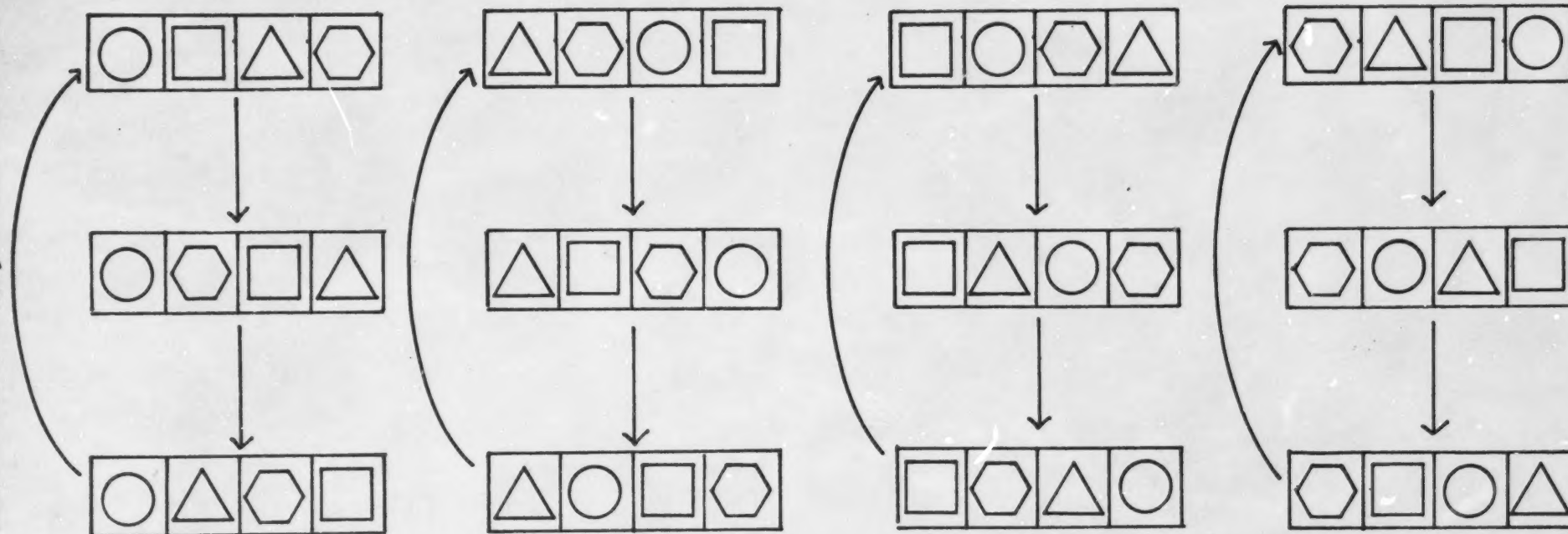
The SECOND MACHINE leaves the shape in the second position where it is and moves the other shapes around as is shown here by the arrows:



If you look at your MACHINE REMINDER sheets you will see how the frames are changed into one another by the two MACHINES.

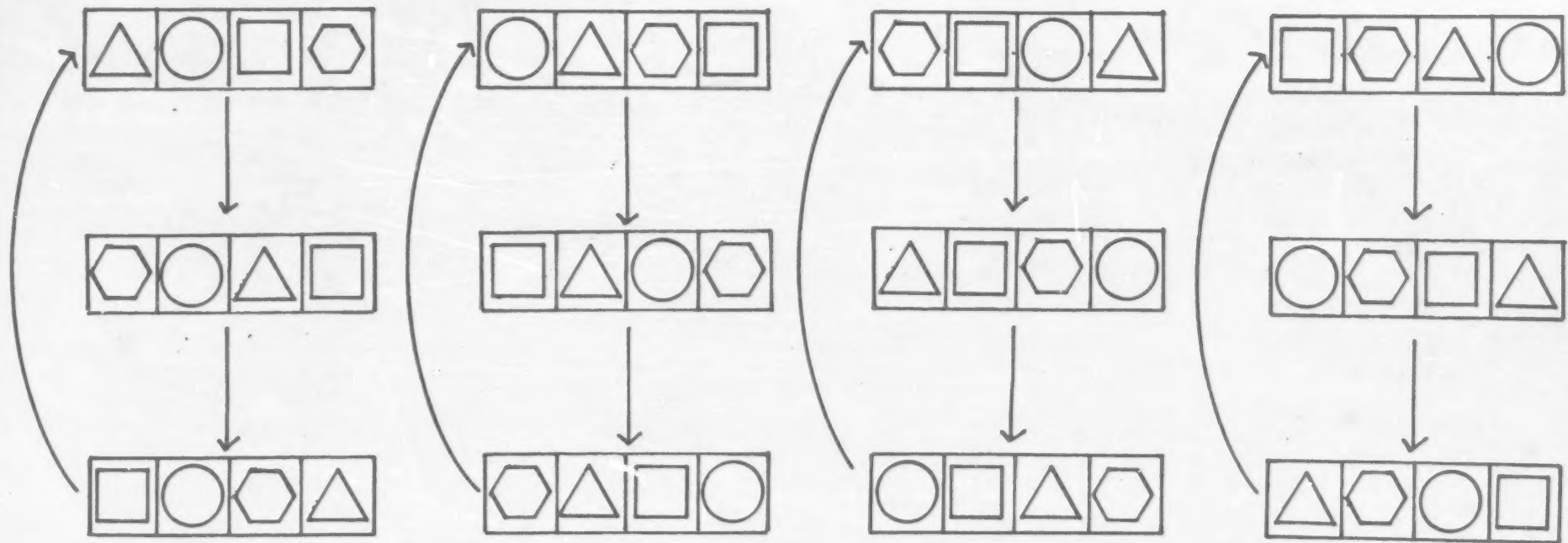
MACHINE REMINDER FOR GAME G

The FIRST MACHINE moves the shapes around like this



MACHINE REMINDER FOR GAME G

The SECOND MACHINE moves the shapes around like this



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MACHINE DESCRIPTION FOR GAME H

In this Game there are two MACHINES, the TIME MACHINE and the SPECIAL MACHINE.

The TIME MACHINE changes the time (tense) of the verb from present to future, from future to past, and from past to present.

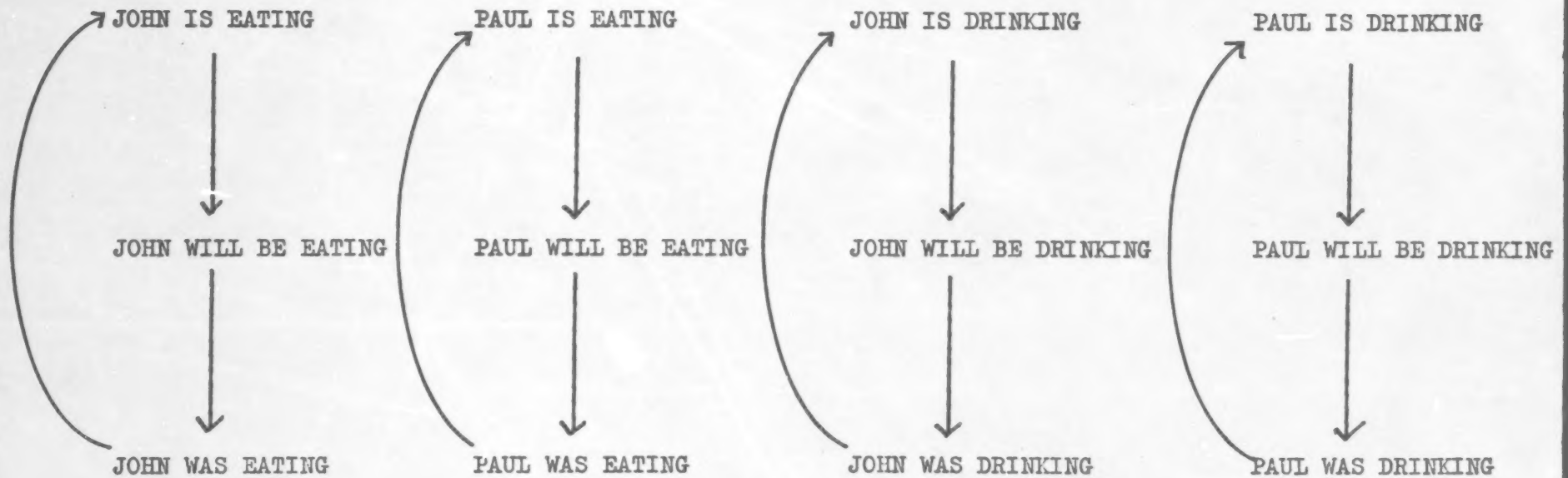
The SPECIAL MACHINE works as follows:

- (1) If the verb is in the present this machine changes the time to the past, the meaning of the verb, and the meaning of the noun.
- (2) If the verb is in the past this machine changes the time to the future, and the meaning of the verb. The noun it does not change.
- (3) If the verb is in the future this machine changes the time to the present, and the meaning of the noun. The meaning of the verb it does not change.

If you look at your MACHINE REMINDER sheets you will see how the sentences are changed into one another by the two MACHINES.

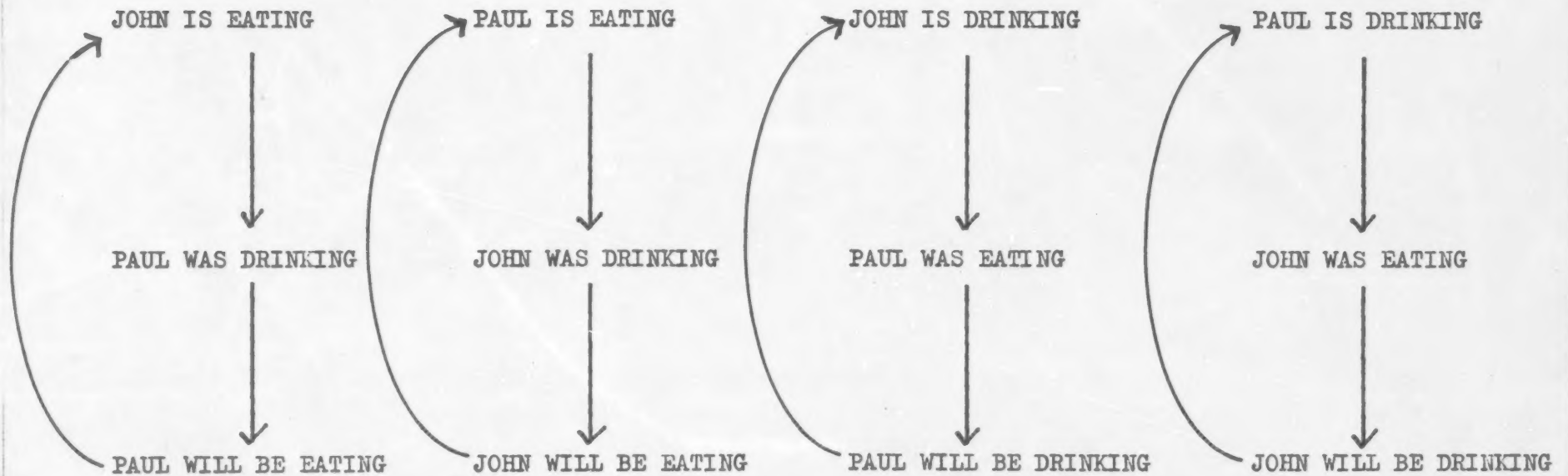
MACHINE REMINDER FOR GAME H

The TIME MACHINE moves the sentences around like this



MACHINE REMINDER FOR GAME H

The SPECIAL MACHINE moves the sentences around like this



- 250 -

SPECIAL
APPENDICES
TO
CHAPTER 4

APPENDIX S4 - 1

The Opening Instructions and Personal Data Sheet.

Opening Instructions

1. Ask students to complete a Personal Data Sheet.
2. Distribute the Initial Teaching Booklet, Machine Description and Machine Reminders for Game B.
Make sure every member of the class has a pencil and a sheet of blank paper.
3. Place the Model of the Tetrahedron on the teacher's desk.
4. Read:

Your class teacher and your principal have kindly agreed to let you spend two of your mathematics classes on learning to work with some new mathematical Games. To-day you will be taught a Game with numbers and in a week from today you will also learn a Game with (alter as necessary) shapes. What you do in both Games will tell us whether you are equally good at both Games, or better at one than at the other. The mathematics in both Games is the same.

Some of you will be quicker at learning the new Games than others. Do not let this worry you. It does not matter how well or badly you answer the questions that you are asked. Your marks in the Games will be given to you and to your class teacher, but they will in no way affect the marks you have earned at school.

The Games are something extra. Relax and enjoy them as much as you can!

5. Hold up the Model of the Tetrahedron.

6. Read:

I have here a model of one of the regular geometric solids. It has 4 corners or vertices, and 4 faces that are all equilateral triangles.

What is its name?

7. Wait 30 seconds for answers.

8. Read:

It is a regular pyramid built on an equilateral triangle. Such a pyramid is called a tetrahedron. This is a long name, but you will be able to answer all the questions even if you do not remember it.

If I had a box into which this tetrahedron fitted exactly, in how many different ways could I put it into this box?

9. Wait 30 seconds for answers.

10. Read:

There are 12 ways in all. Let us see why. The vertex I am now holding in my hand could be placed into any of the 4 corners of the box. This makes 4 choices. What about the triangle opposite this vertex? Can you see from the way I am turning it that it could take up 3 different positions in the box?

11. Wait 30 seconds for answers.

12. Read:

So after the 4 choices for the vertex there are 3 choices for the triangle opposite this vertex. This makes $4 \times 3 = 12$ different ways in all.

13. Place the Model of the Tetrahedron on the teacher's desk and hold up one of the Machine Reminder charts.

14. Read:

You all have two charts like this one. They are called Machine Reminders. On each of them are printed 12 different numbers. These numbers represent the 12 positions of the tetrahedron. In a few moments we will start reading the Initial Teaching Booklet. For this you will learn how to change any one of the numbers into any other number by using one or both of two machines.

Are you ready?

15. Wait 30 seconds for answers.

16. Begin reading through the Initial Teaching Booklet in accordance with its directives.

Personal Data Sheet

Please PRINT your answers.

Last Name:

First Name:

Age: (Give years and months)

Name of School:

Grade:

Boy or Girl:

(Wherever possible the tester should add the I.Q. score
to this sheet later.)

APPENDIX S4 - 2

The Task Analysis
of the
Initial Learning Test
for
Game B

In each task the part which subjects were asked to complete has been placed between asterisks.

Question No.	Starting Position	Machine Operation(s)	End Position
(1)	5	Two-Four	* 10 *
(2)	2	Switch	* 10 *
(3)	30	* Two-Four *	60
(4)	4	* Switch *	60
(5)	20	Switch/Two-Four	* 3 *
(6)	60	Switch/Switch	* 60 *
(7)	10	*Two-Four/Switch*	12
(8)	1	*Switch/Two-Four*	6
(9)	3	*Switch/Two-Four, Switch*	10
(10)	12	*Two-Four/Two-Four/Switch/Two-Four*	60

APPENDIX S4 - 3

The Introduction to the Transfer Test

between

Game B and Game C

TRANSFER QUESTIONS

BETWEEN

GAME B AND GAME C

In each question you will see either black lines _____

or empty frames

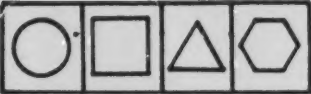
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or both.

Please try to find the correct shapes for the empty frames,
and the correct numbers or names of machines for the black lines.

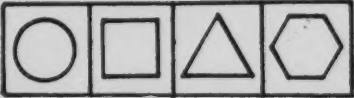
Please look carefully at your MACHINE REMINDERS for Game B and Game C
as you try to answer each question.

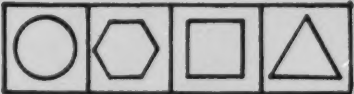
To help you match up the MACHINE REMINDERS for these two Games correctly
here are three rules:

The frame  in Game C matches up with the number 1 in Game B.

The FIRST MACHINE in Game C matches up with the TWO-FOUR MACHINE in Game B.

Look at your MACHINE REMINDERS.

If in Game C you go from this frame → 

to this frame → 

by using the FIRST MACHINE

then in Game B you go from this number → 1

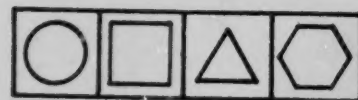
to this number → 2

by using the TWO-FOUR MACHINE.

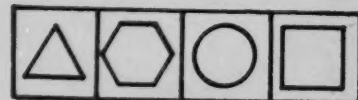
The SWITCH MACHINE in Game C matches up with the SWITCH MACHINE in Game B.

Look at your MACHINE REMINDERS.

If in Game C you go from this frame



to this frame



by using the SWITCH MACHINE

then in GAME B you go from this number

1

to this number

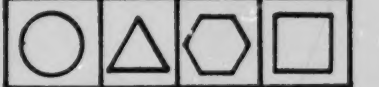
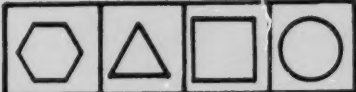
3

by using the SWITCH MACHINE.

APPENDIX S4 - 4

The Task Analysis for the Transfer Test
between
Game B and Game C

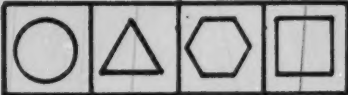
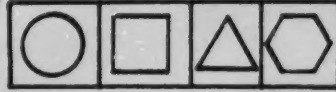
In each task the part which subjects were asked to complete has been placed between asterisks.

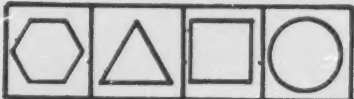
<u>Question No.</u>	<u>Starting Position</u>	<u>Machine Operation(s)</u>	<u>Final Position</u> *
(1)	 2	FIRST TWO-FOUR	 * 4 *
(2)	15 	SWITCH SWITCH	* 5 *  * * *

<u>Question No.</u>	<u>Starting Position</u>	<u>Machine Operation(s)</u>	<u>Final Position</u>
(3)	 3	FIRST/FIRST TWO-FOUR/TWO-FOUR	 * 12 *
(4)	5 	SWITCH/TWO-FOUR SWITCH/FIRST	* 30 * 

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<u>Question No.</u>	<u>Starting Position</u>	<u>Machine Operation(s)</u>	<u>Final Position</u>
(5)	 4	* FIRST/SWITCH * * TWO-FOUR/SWITCH *	 * 3 *
(6)	60 	* SWITCH/TWO-FOUR * * SWITCH/FIRST *	1 

<u>Question No.</u>	<u>Starting Position</u>	<u>Machine Operation(s)</u>	<u>Final Position</u>
(7)	 12	*SWITCH/FIRST/SWITCH* *SWITCH/TWO-FOUR/SWITCH *	 * 15 *
(8)	30 	*SWITCH/TWO-FOUR/TWO-FOUR* *SWITCH/FIRST/FIRST*	3 

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<u>Question No.</u>	<u>Starting Position</u>	<u>Machine Operation(s)</u>	<u>Final Position</u>
(9)	 6	*FIRST/SWITCH/FIRST/FIRST* * TWO-FOUR/SWITCH TWO-FOUR/TWO-FOUR *	 * 10 *
(10)	10 	* TWO-FOUR/TWO-FOUR SWITCH/TWO-FOUR * *FIRST/FIRST/SWITCH/FIRST*	30 

APPENDIX S4 - 5

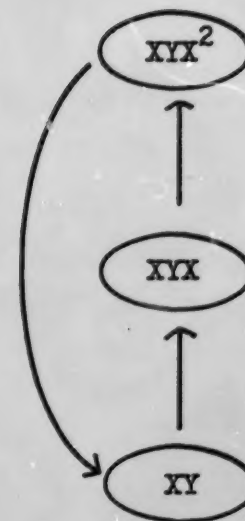
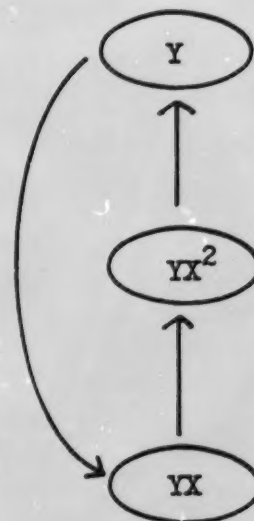
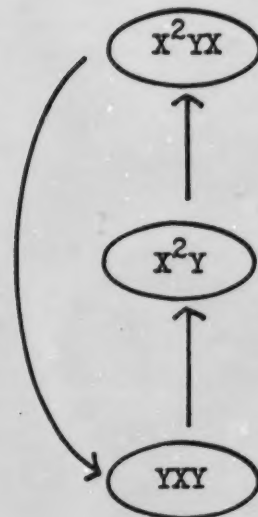
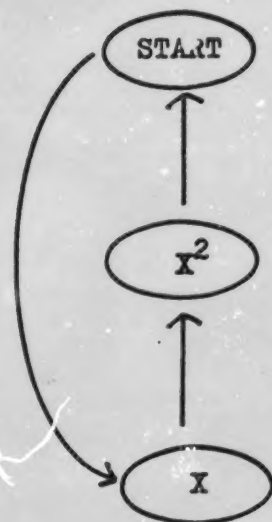
The Permuted Machine Reminder Charts for
Game A, Game C, and Game D.

Row Perm (1)(23)

MACHINE REMINDER FOR GAME A

Col Perm (1)(234)

The X MACHINE moves the letters around like this

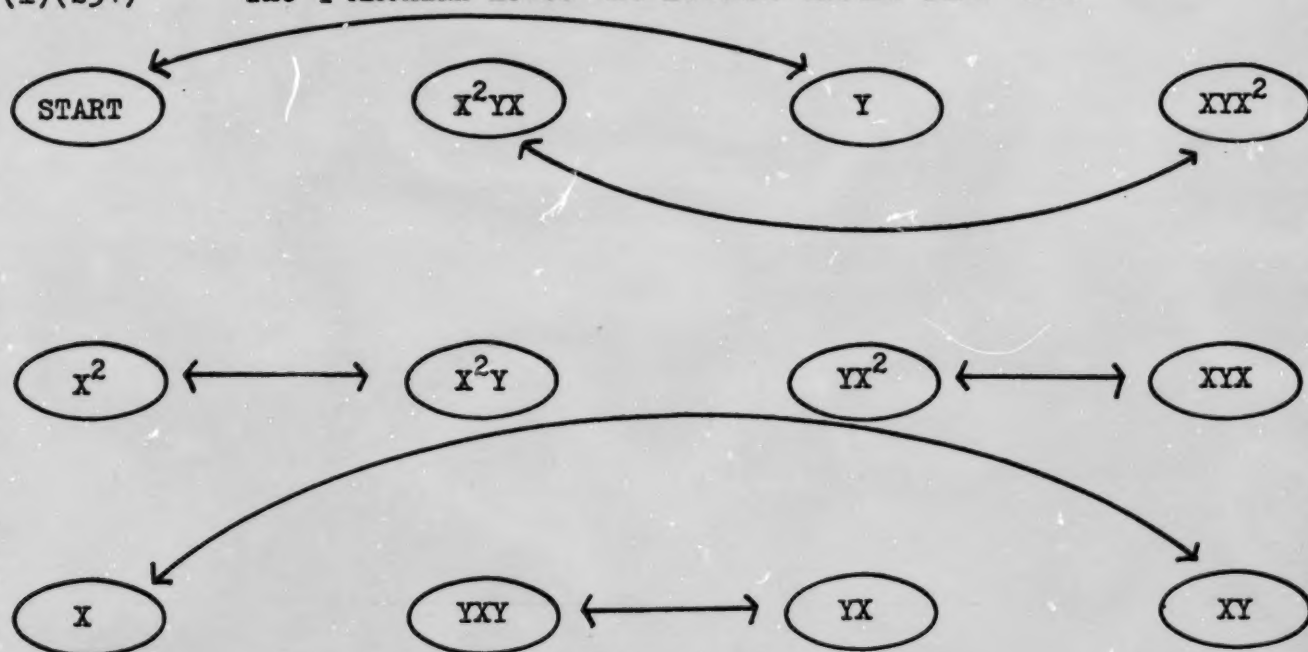


Perm (1)(23)

MACHINE REMINDER FOR GAME A

Perm (1)(234)

The Y MACHINE moves the letters around like this

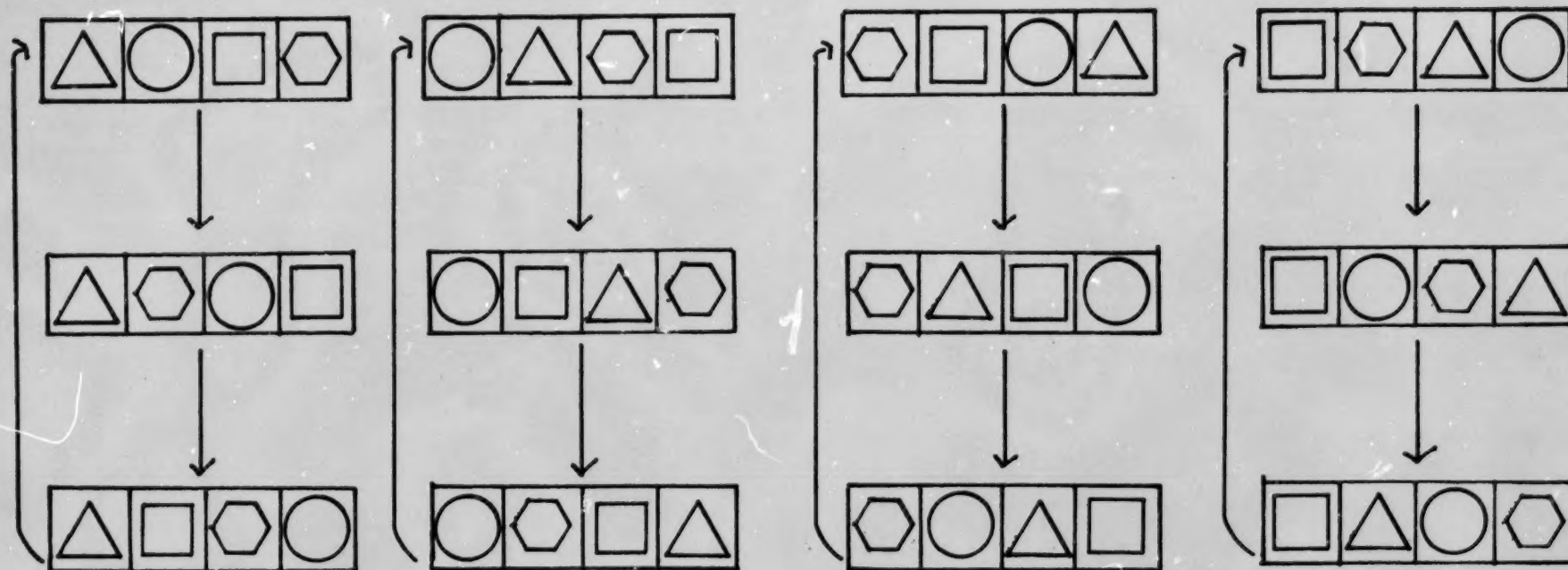


Row Perm (123)

Col Perm (12)(34)

MACHINE REMINDER FOR GAME C

The FIRST MACHINE moves the shapes around like this

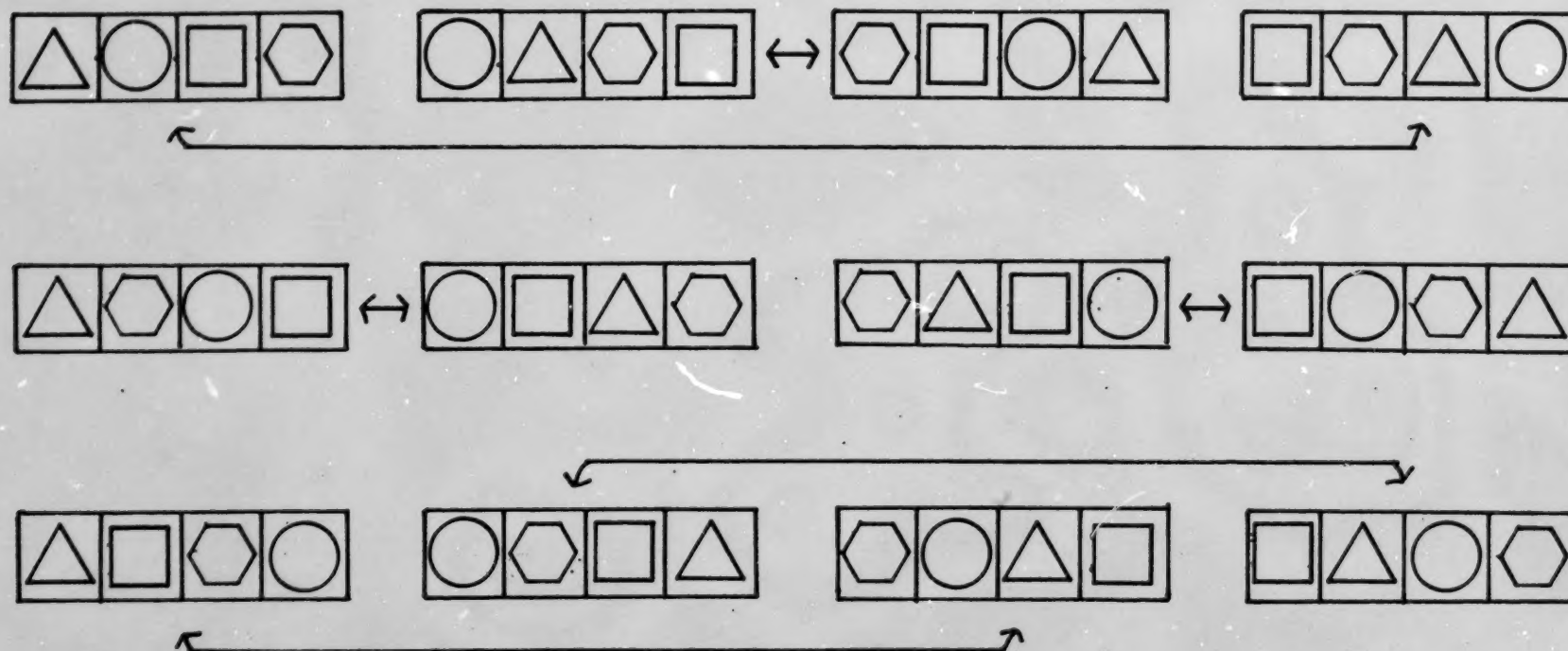


Row Perm (123)

Col Perm (12)(34)

MACHINE REMINDER FOR GAME C

The SWITCH MACHINE moves the shapes around like this



310

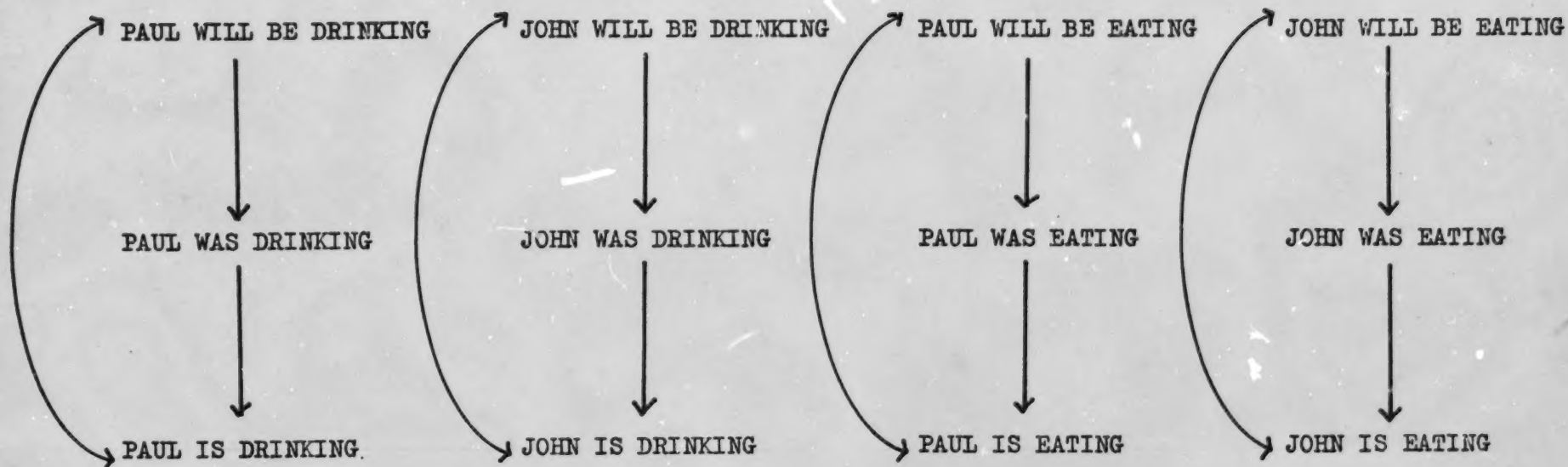
311

Row Perm (123)

Col Perm (14)(23)

MACHINE REMINDER FOR GAME D

The TIME MACHINE moves the sentences around like this



312

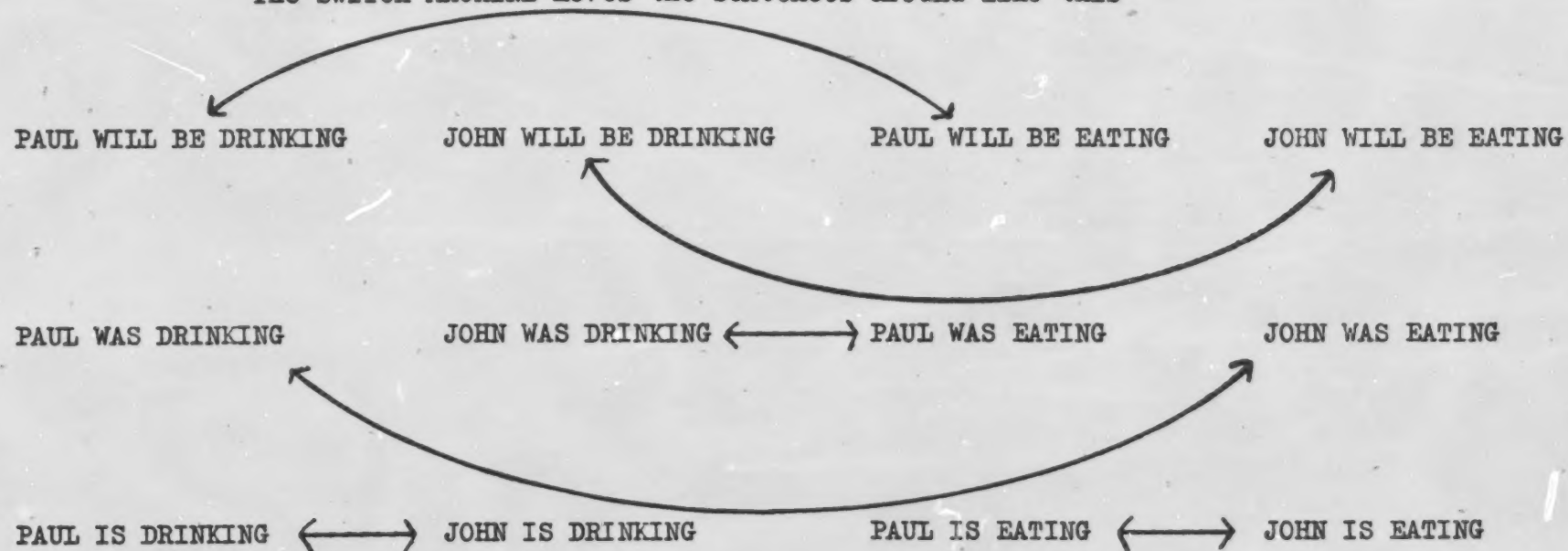
313

Row Perm (123)

Col Perm (14)(23)

MACHINE REMINDER FOR GAME D

The SWITCH MACHINE moves the sentences around like this



APPENDIX S4 - 6

The Introduction to the Transfer Test
between
Game B and Game F

TRANSFER QUESTIONS

BETWEEN

GAME B AND GAME F

In each question you will see empty black lines _____.

Please try to find the correct names of the machines
for the empty black lines.

Please look carefully at your MACHINE REMINDERS for Game B and Game F
as you try to answer each question.

To help you match up the MACHINE REMINDERS for these two Games correctly,
here are four rules:

- (1) The numbers in Game B are the same as the numbers in Game F.
- (2) The TWO-FOUR MACHINE in Game B is also the TWO-FOUR MACHINE in Game F. Look at your MACHINE REMINDEKS.

If in Game B you go from this number —————→ 1

to this number —————→ 2

by using the TWO-FOUR MACHINE

then in Game F you go from this number —————→ 1

to this number —————→ 2

by using the TWO-FOUR MACHINE

The SWITCH MACHINE in Game B does NOT match up with the
SPECIAL MACHINE in Game F.

Look at your MACHINE REMINDERS.

If in Game B you go from this number _____→ 1

to this number _____→ 3

by using the SWITCH MACHINE

then in Game F you go from this number _____→ 1

to this number _____→ 3

by using the TWO-FOUR MACHINE
and then the SPECIAL MACHINE.

The SPECIAL MACHINE in Game F does NOT match up with the SWITCH MACHINE in Game B.

Look at your MACHINE REMINDERS.

If in Game F you go from this number $\longrightarrow$ 1
to this number $\longrightarrow$ 60

by using the SPECIAL MACHINE

then in Game B you go from this number $\longrightarrow$ 1
to this number $\longrightarrow$ 60

by using the TWO-FOUR MACHINE

and then the TWO-FOUR MACHINE

and then the SWITCH MACHINE.

So you see that the number of machines you have to use to go from one number to another is not always the same in the two Games.

APPENDIX S4 - 7

The Task Analysis for the Transfer Test

between

Game B and Game F

In each task the part which subjects were asked to complete has been placed between asterisks.

<u>Question No.</u>	<u>Starting Position</u>	<u>Machine Operation(s)</u>	<u>Final Position</u>
(1)	2	* SWITCH *	10
	2	TWO-FOUR/*SPECIAL*	10
(2)	15	* SPECIAL *	4
	15	TWO-FOUR/TWO-FOUR/*SWITCH*	4

<u>Question No.</u>	<u>Starting Position</u>	<u>Machine Operation(s)</u>	<u>Final Position</u>
(3)	3	*SWITCH/TWO-FOUR*	2
	3	*SPECIAL/SPECIAL*	2
(4)	5	TWO-FOUR/*TWO-FOUR/SPECIAL*	2
	5	*TWO-FOUR/SWITCH*	2

<u>Question No.</u>	<u>Starting Position</u>	<u>Machine Operation(s)</u>	<u>Final Position</u>
(5)	4	SWITCH/*TWO-FOUR/TWO-FOUR*	30
	4	SPECIAL/*SPECIAL/TWO-FOUR*	30
(6)	60	SPECIAL/*SPECIAL/SPECIAL*	60
	60	*SWITCH/SWITCH*	60

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<u>Question No.</u>	<u>Starting Position</u>	<u>Machine Operation(s)</u>	<u>Final Position</u>
(7)	12	*TWO-FOUR/SWITCH/TWO-FOUR*	2
	12	*TWO-FOUR/SPECIAL/SPECIAL*	2
(8)	30	*SPECIAL/TWO-FOUR/TWO-FOUR*	20
	30	*SWITCH/TWO-FOUR/SWITCH*	20

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<u>Question No.</u>	<u>Starting Position</u>	<u>Machine Operation(s)</u>	<u>Final Position</u>
(9)	10	*TWO-FOUR/SWITCH/ TWO-FOUR/TWO-FOUR*	30
	10	*SPECIAL/TWO-FOUR*	30
(10)	6	*SPECIAL/TWO-FOUR/ TWO-FOUR/SPECIAL*	10
	6	*TWO-FOUR/SWITCH/ TWO-FOUR/TWO-FOUR*	10

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SPECIAL APPENDICES

TO

CHAPTER 5

SPECIAL APPENDIX S5 - 1
The Mean Residual Scores.

	C_2	C_3	C_4	MIN_{1234}	MIN_{12}
Total group:					
Maximum	5.66	7.82	6.60	5.94	4.70
Minimum	-9.30	-7.69	-8.77	-7.97	-9.29
Mean	0.03	-0.05	0.04	-0.05	0.04
Ranked Mean	(7) 1.98	(3) 1.01	(7) 1.52	(7) 2.18	(7) 1.82
Residuals	(8) 0.61	(7) 0.58	(3) 0.46	(6) 0.40	(3) 0.49
of the Treat- ment	(2) 0.42	(1) 0.33	(6) 0.44	(3) 0.32	(2) 0.46
Groups (1)	(3) 0.42	(2) 0.26	(4) 0.20	(2) 0.14	(8) 0.37
to (8)	(4) -0.16	(4) 0.23	(1) 0.15	(4) 0.02	(4) -0.04
	(1) -0.24	(6) -1.00	(2) 0.03	(1) -0.47	(1) -0.23
	(6) -1.10	(8) -1.05	(6) -0.93	(6) -0.91	(6) -0.96
	(5) -1.17	(5) -1.07	(5) -1.49	(5) -1.66	(5) -1.18

(continued on next page)

(Continued from p. 287)

	MIN ₃₄	MIN ₁₃	MAX ₁₂₃₄	MAX ₁₂	MAX ₃₄
Total group:					
Maximum	7.04	4.12	6.13	4.69	7.45
Minimum	-8.65	-7.48	-6.36	-6.18	-7.93
Mean	-0.02	0.02	0.01	-0.02	-0.05
Ranked Mean	(7) 1.41	(7) 0.72	(3) 0.47	(8) 0.21	(3) 0.79
Residuals of	(3) 0.65	(3) 0.54	(1) 0.23	(7) 0.14	(7) 0.63
the Treatment	(4) 0.25	(2) 0.45	(4) 0.07	(1) 0.00	(1) 0.26
Groups (1) to	(1) 0.09	(4) 0.22	(5) -0.02	(5) -0.01	(2) 0.21
(8).	(2) 0.05	(1) 0.15	(7) -0.04	(2) -0.06	(4) 0.13
	(8) -0.10	(6) -0.45	(8) -0.12	(4) -0.08	(8) -0.56
	(6) -0.97	(8) -0.74	(2) -0.15	(3) -0.14	(6) -1.00
	(5) -1.54	(5) -0.85	(6) -0.51	(6) -0.15	(5) -1.05

(continued on next page)

(Continued from p.288.)

	MAX ₁₃	D ₁₂₃₄	D ₁₂	D ₃₄	D ₁₃
Total group:					
Maximum	6.41	7.77	9.33	8.61	7.84
Minimum	-2.95	-6.00	-4.77	-3.89	-4.02
Mean	0.02	-0.02	0.04	0.05	0.03
Ranked Mean	(3) 0.57	(5) 1.50	(5) 1.23	(5) 0.56	(5) 0.75
Residuals of	(1) 0.27	(1) 0.60	(6) 0.85	(1) 0.27	(8) 0.55
the Treatment	(4) 0.10	(6) 0.28	(1) 0.31	(3) 0.25	(1) 0.17
Groups (1) to	(7)-0.05	(3) 0.23	(4) 0.05	(2) 0.18	(3) 0.06
(8).	(2)-0.11	(4) -0.04	(8) -0.07	(6) 0.08	(6) 0.01
	(5)-0.14	(2) -0.40	(3) -0.38	(4) -0.02	(2) -0.09
	(8)-0.22	(8) -0.63	(2) -0.47	(8) -0.37	(2) -0.53
	(6)-0.48	(7) -2.30	(7) -1.60	(7) -0.69	(7) -0.77

SPECIAL APPENDIX S5 - 2

Tables of Values of p less than 0.010 obtained from
Mann Whitney U tests between Treatment pairs on the
Residual Scores.

Table A.

p = 0.000

C_2	C_3	C_4	MIN_{1234}	MIN_{12}
(3,6)	(3,5)	(5,7)	(1,7)	(3,6)
(1,7)	(3,6)	(6,7)	(2,7)	(1,7)
(3,7)	(3,8)		(3,7)	(3,7)
(4,7)	(7,8)		(4,7)	(4,7)
(5,7)			(5,7)	(5,7)
(6,7)			(6,7)	(6,7)
(6,8)			(7,8)	(7,8)
(7,8)				

MIN_{34}	MIN_{13}	MAX_{1234}	MAX_{12}	MAX_{34}
(5,7)	(7,8)		(1,3)	(3,5)
(6,7)			(3,5)	(3,6)
			(1,6)	(6,7)
		NONE	(5,6)	
			(3,7)	
			(6,7)	
			(3,8)	

(continued on page 291)

(continued from page 290)

MAX_{13}	D_{1234}	D_{12}	D_{34}	D_{13}
NONE	(1,7) (2,7) (3,7) (5,7) (6,7) (7,8)	(4,7) (5,7) (6,7) (7,8)	NONE	(3,7)

Table B.

$$p = 0.001$$

C_2	C_3	C_4	MIN_{1234}	MIN_{12}
(2,6)	(5,7)	NONE	(3,5)	(2,6)
(2,7)	(6,7)			(2,7)
	(1,8)			

MIN_{34}	MIN_{13}	MAX_{1234}	MAX_{12}	MAX_{34}
(7,8)	(3,5)	NONE	NONE	(3,8)
	(5,7)			(5,7)

MAX_{13}	D_{1234}	D_{12}	D_{34}	D_{13}
(1,6)	(4,7)	(1,7)	NONE	(5,7)
(3,6)		(3,7)		

Table C.

 $0.002 \leq p < 0.005$

C_2	C_3	C_4	MIN_{1234}	MIN_{12}
(4,6)	(1,5) (1,6) (4,8)	(3,5) (4,5) (1,7) (5,8) (6,8) (7,8)	(5,8)	(4,6) (6,8)

MIN_{34}	MIN_{13}	MAX_{1234}	MAX_{12}	MAX_{34}
(3,5) (3,6) (4,6)	(3,6) (6,7) (3,8)	(1,6) (3,6)	(3,4) (4,7) (6,8)	(1,5) (1,6) (7,8)

MAX_{13}	D_{1234}	D_{12}	D_{34}	D_{13}
(1,8)	(5,8)	(2,6) (3,6)	NONE	NONE

Table D.

$$0.005 \leq p \leq 0.010$$

C_2	C_3	C_4	MIN_{1234}	MIN_{12}
NONE	(3,4) (4,6)	(1,5) (3,6) (4,6) (2,7)	(2,5) (4,5) (3,6) (6,8)	(3,5)

MIN_{34}	MIN_{13}	MAX_{1234}	MAX_{12}	MAX_{34}
(1,5) (1,7) (4,5)	(1,6)	NONE	(2,7)	(4,6)

MAX_{13}	D_{1234}	D_{12}	D_{34}	D_{13}
NONE	(2,5)	NONE	NONE	(2,5) (7,8)

Table E.

The number of p values less than 0.010 observed
for each Treatment at the levels given in Tables A to D.

Values of p	Treatment No.							
	1	2	3	4	5	6	7	8
Table A	6	2	16	4	11	16	39	10
Table B	3	4	5	1	6	5	11	3
Table C	7	1	8	7	8	15	5	11
Table D	4	5	4	6	8	7	4	2
Total	20	12	33	18	33	43	59	26

Estimates of the Type I Error in the Multiple Comparisons of Section C if all p values below a fixed Rejection Level are judged significant.

The risk of falsely judging at least one p value significant and committing a Type I Error at a chosen level of rejection was calculated as the sum of the contributions to this error of all the hypotheses that would in fact be rejected at this level. Since the acceptable risk may vary from one experiment to the next, the accumulated probability of a Type I Error has been listed in Table F for rejection levels from 0.000(25) to 0.010.

The computer print-outs gave p values to 3 places of decimals. Each 0.000 value was therefore increased by 0.00025, the mean value of the uncomputed probability, and every value was multiplied by the number of times it was observed in the results. The number obtained in this way was the contribution of this value to the Type I Error.

Table F
Estimates of the Type I Error

Value of p	No. of Observations	Contribution to Type I Error	Cumulative Type I Error
0.000(25)	52	0.013	0.013
0.001	19	0.019	0.032
0.002	13	0.026	0.058
0.003	7	0.021	0.079
0.004	11	0.044	0.123
0.005	3	0.015	0.138
0.006	6	0.036	0.174
0.007	5	0.035	0.209
0.008	2	0.016	0.225
0.009	4	0.036	0.261
0.010	3	0.030	0.291

SPECIAL APPENDIX S5 - 3

The Raw Mean Scores of the Treatment Groups.

TABLE A

Treatment Group No. 1

Description: Transfer from Game B to Game C. Permuted Machine Reminders for Game C. 86 Subjects from Grades 7 to 9. Mean IQ score 115. Mean IL score 17.

Columns	C ₁ 10.5	C ₂ 8.1	C ₃ 10.2	C ₄ 8.7
Equivalence	MIN ₁₂₃₄ 6.7	MIN ₁₂ 8.0	MIN ₃₄ 8.4	MIN ₁₃ 9.4
Best Performance	MAX ₁₂₃₄ 11.3	MAX ₁₂ 10.6	MAX ₃₄ 10.4	MAX ₁₃ 11.3
Stress	D ₁₂₃₄ 4.6	D ₁₂ 2.6	D ₃₄ 2.0	D ₁₃ 1.9

Table B

Treatment Group No.2

Description: Transfer from Game B to Game C. Permuted Machine Reminders for Game C. 39 Subjects from Grades 10 to 12. Mean IQ score 107. Mean IL score 17.

Columns	C ₁ 8.9	C ₂ 7.0	C ₃ 8.9	C ₄ 6.9
Equivalence	MIN ₁₂₃₄ 5.6	MIN ₁₂ 7.0	MIN ₃₄ 6.7	MIN ₁₃ 8.0
Best Performance	MAX ₁₂₃₄ 9.9	MAX ₁₂ 9.0	MAX ₃₄ 9.1	MAX ₁₃ 9.8
Stress	D ₁₂₃₄ 4.3	D ₁₂ 2.0	D ₃₄ 2.4	D ₁₃ 1.8

Table C

Treatment Group No. 3

Description: Transfer from Game B to Game C. Machine Reminders for Game C without Permutations 64 Subjects from Grades 7 to 9. Mean IQ score 107. Mean IL score 14.

Columns	C ₁ 7.7	C ₂ 5.6	C ₃ 8.5	C ₄ 6.0
Equivalence	MIN ₁₂₃₄ 4.4	MIN ₁₂ 5.5	MIN ₃₄ 5.9	MIN ₁₃ 6.7
Best Performance	MAX ₁₂₃₄ 9.6	MAX ₁₂ 7.8	MAX ₃₄ 8.6	MAX ₁₃ 9.5
Stress	D ₁₂₃₄ 5.2	D ₁₂ 2.3	D ₃₄ 2.7	D ₁₃ 2.8

Table D

Treatment Group No. 4

Description: Transfer from Game B to Game C. Machine Reminders for Game B as always. Machine Reminders for Game C showed only the state in the top left corner, had all other frames empty and showed no transformation lines.
54 Subjects from Grades 10 to 12. Mean IQ score 119. Mean IL score 18.5.

Columns	C ₁ 10.8	C ₂ 8.9	C ₃ 10.6	C ₄ 9.4
Equivalence	MIN ₁₂₃₄ 7.9	MIN ₁₂ 8.9	MIN ₃₄ 9.3	MIN ₁₃ 10.0
Best Performance	MAX ₁₂₃₄ 11.3	MAX ₁₂ 10.8	MAX ₃₄ 10.7	MAX ₁₃ 11.3
Stress	D ₁₂₃₄ 3.4	D ₁₂ 1.9	D ₃₄ 1.4	D ₁₃ 1.3

Table E

Treatment Group No. 5

Description: Transfer from Game B to Game A. Permuted Machine Reminders for Game A. 51 Subjects from Grades 10 to 12. Mean IQ score 110. Mean IL score 17.5 .

Columns	C ₁ 10.7	C ₂ 7.1	C ₃ 8.9	C ₄ 6.9
Equivalence	MIN ₁₂₃₄ 5.4	MIN ₁₂ 7.0	MIN ₃₄ 6.6	MIN ₁₃ 8.5
Best Performance	MAX ₁₂₃₄ 11.2	MAX ₁₂ 10.8	MAX ₃₄ 9.1	MAX ₁₃ 11.1
Stress	D ₁₂₃₄ 5.8	D ₁₂ 3.8	D ₃₄ 2.5	D ₁₃ 2.6

Table F

Treatment Group No. 6

Description: Transfer from Game B to Game D. Permuted Machine Reminders for Game D. 67 Subjects from Grades 7 to 9. Mean IQ score 102. Mean IL score 14.

Columns	C ₁ 7.5	C ₂ 3.7	C ₃ 6.2	C ₄ 4.2
Equivalence	MIN ₁₂₃₄ 2.6	MIN ₁₂ 3.6	MIN ₃₄ 3.8	MIN ₁₃ 5.4
Best Performance	MAX ₁₂₃₄ 8.4	MAX ₁₂ 7.5	MAX ₃₄ 6.6	MAX ₁₃ 8.3
Stress	D ₁₂₃₄ 5.8	D ₁₂ 3.9	D ₃₄ 2.8	D ₁₃ 3.0

Table G

Treatment Group No. 7

Description: Transfer from Game B to Game F. Games B and F of different type. 54 Subjects from Grades 10 to 12. Mean IQ score 117. Mean IL score 19.

Columns	C ₁	C ₂	C ₃	C ₄
	11.5	11.5	11.4	11.1
Equivalence	MIN ₁₂₃₄ 10.5	MIN ₁₂ 11.3	MIN ₃₄ 10.9	MIN ₁₃ 11.1
Best Performance	MAX ₁₂₃₄ 11.8	MAX ₁₂ 11.8	MAX ₃₄ 11.6	MAX ₁₃ 11.8
Stress	D ₁₂₃₄ 1.3	D ₁₂ 0.5	D ₃₄ 0.8	D ₁₃ 0.6

Table H

Treatment Group No. 8

Description: Transfer from Game B to Game G. Games B and G of different type. 45 Subjects from Grades 10 to 12. Mean IQ score 117. Mean IL score 18.

Columns	C ₁	C ₂	C ₃	C ₄
	10.8	9.5	9.2	9.5
Equivalence	MIN ₁₂₃₄ 8.1	MIN ₁₂ 9.2	MIN ₃₄ 8.8	MIN ₁₃ 9.0
Best Performance	MAX ₁₂₃₄ 11.2	MAX ₁₂ 11.2	MAX ₃₄ 9.9	MAX ₁₃ 11.0
Stress	D ₁₂₃₄ 3.1	D ₁₂ 2.0	D ₃₄ 1.1	D ₁₃ 2.0

GENERAL APPENDICES

APPENDIX G1

THE INITIAL TEACHING BOOKLET

FOR

GAME B

Tetrahedral Group
Factors of 60
One 3-cycle machine
One 2-cycle machine

Initial Teaching Booklet

G A M E B

TEACHING AND EXERCISES

In this game you will be playing with 12 numbers.

They are the numbers

1 2 3 4 5 6 10 12 15 20 30 60

Each of these numbers divides into 60 without leaving a remainder.

For example,

15 divides into 60 to give 4,

and

12 divides into 60 to give 5.

355

You will be moving these numbers around by using two machines.

One of these machines is called the TWO-FOUR MACHINE.

The other machine is called the SWITCH MACHINE.

Let us see how these machines work.

The TWO-FOUR MACHINE works like this:

If a number can be divided by 4 then the TWO-FOUR MACHINE divides it by 4.

For example,

20 divides by 4.

Therefore, if you put 20 into the TWO-FOUR MACHINE, you will get 5.

If a number can NOT be divided by 4, then the TWO-FOUR MACHINE multiplies it by 2.

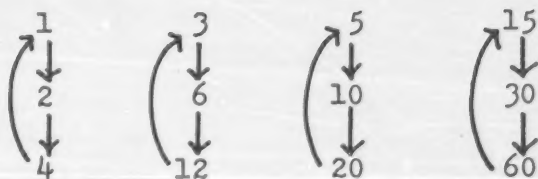
For example,

6 does NOT divide by 4.

Therefore, if you put 6 into the TWO-FOUR MACHINE, you will get 12.

Here are the 12 numbers of your game arranged in four columns.

The arrows point to the number into which each number is changed by the TWO-FOUR MACHINE.



These numbers and arrows you can also find on the sheet called MACHINE REMINDER.

The second machine is called the SWITCH MACHINE.

This machine switches the numbers around in pairs.

Here are the numbers of your game arranged in three rows:

ROW 1	1	3	5	15
-------	---	---	---	----

These are the odd numbers in your game

ROW 2	2	6	10	30
-------	---	---	----	----

These are the odd numbers multiplied by 2

ROW 3	4	12	20	60
-------	---	----	----	----

These are the odd numbers multiplied by 4

To the numbers in ROW 1 the SWITCH MACHINE does this:

$1 \longleftrightarrow 3$ $5 \longleftrightarrow 15$

It multiplies or divides each number by 3, whichever is possible.

For example,

3 divides by 3.

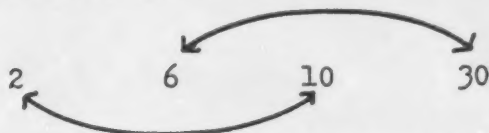
Therefore, if you put 3 into the SWITCH MACHINE, you will get 1.

For example,

5 does NOT divide by 3.

Therefore, if you put 5 into the SWITCH MACHINE, you will get 15.

To the numbers in ROW 2 the SWITCH MACHINE does this:



It multiplies or divides each number by 5, whichever is possible.

For example,

10 divides by 5.

Therefore, if you put 10 into the SWITCH MACHINE, you will get 2.

For example,

6 does NOT divide by 5.

Therefore, if you put 6 into the SWITCH MACHINE, you will get 30.

To the numbers in ROW 3 the SWITCH MACHINE does this:



It multiplies or divides by 3, and it ALSO multiplies or divides by 5, whichever is possible.

For example,

4 does NOT divide by 3 or by 5.

Therefore, if you put 4 into the SWITCH MACHINE, you will get 60.

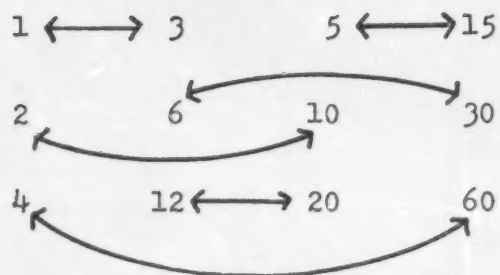
For example,

20 divides by 5 but does NOT divide by 3.

Therefore, if you put 20 into the SWITCH MACHINE, you will get 12.

Here are the 12 numbers of your game arranged in three rows.

The arrows point to the number into which each number is changed by the SWITCH MACHINE.



These numbers and arrows you can also find on the sheet called MACHINE REMINDER.

Now that the two machines have been explained to you, let us try some exercises.
When trying to answer any question you may always look at your MACHINE REMINDER
sheet.

Exercise 1.

If you start with this number  20
and put it into a TWO-FOUR MACHINE
which number will you obtain ?

See whether you can write the answer on your sheet of paper before 1 minute is up.
Do NOT turn to the next page.

When the minute is up the teacher will read the next page with you.

If you start with this number —————→ 20

and put it into a TWO-FOUR MACHINE

you will obtain this number —————→ 5

Your number is 20. This number divides by 4.

Therefore the TWO-FOUR MACHINE changes the number into 5.

$$20 \div 4 = 5$$

Exercise 2.

If you start with this number $\longrightarrow$ 6
and put it into a TWO-FOUR MACHINE
which number will you obtain ?

See whether you can write the answer on your sheet of paper before 1 minute is up.
Do NOT turn to the next page.

When the minute is up the teacher will read the next page with you.

If you start with this number $\longrightarrow$ 6

and put it into a TWO-FOUR MACHINE

you will obtain this number $\longrightarrow$ 12

Your number is 6. This number does NOT divide by 4.

Therefore the TWO-FOUR MACHINE changes the number into 12.

$$6 \times 2 = 12$$

Exercise 3.

If you start with this number $\longrightarrow$ 10
and put it into a SWITCH MACHINE
which number will you obtain ?

See whether you can write the answer on your sheet of paper before 1 minute is up.
Do NOT turn to the next page.

When the minute is up the teacher will read the next page with you.

If you start with this number —————→ 10

and put it into a SWITCH MACHINE

you will obtain this number —————→ 2

Your number is 10. This number divides by 2 but NOT by 4.

Therefore the SWITCH MACHINE changes the number into 2.

$$10 \div 5 = 2$$

Exercise 4.

If you start with this number $\longrightarrow$ 20
and put it into a SWITCH MACHINE
which number will you obtain ?

See whether you can write the answer on your sheet of paper before 1 minute is up.

Do NOT turn to the next page.

When the minute is up the teacher will read the next page with you.

If you start with this number $\longrightarrow$ 20

and put it into a SWITCH MACHINE

you will obtain this number $\longrightarrow$ 12

Your number is 20. This number divides by 4.

Therefore the SWITCH MACHINE changes the number into 12.

$$20 \times 3 = 60, \text{ and } 60 \div 5 = 12$$

$$20 \div 5 = 4, \text{ and } 4 \times 3 = 12$$

Exercise 5.

If you start with this number	—————→	60
and wish to obtain this number	—————→	15

which machine will you use?

See whether you can write the answer on your sheet of paper before 1 minute is up.

Do NOT turn to the next page.

When the minute is up the teacher will read the next page with you.

If you start with this number $\longrightarrow$ 60
and wish to obtain this number $\longrightarrow$ 15
you must use a TWO-FOUR MACHINE.

To obtain 15 from 60 you must divide 60 by 4.

The machine which divides by 4 is the TWO-FOUR MACHINE.

Exercise 6.

If you start with this number $\longrightarrow$ 12
and wish to obtain this number $\longrightarrow$ 20
which machine will you use?

See whether you can write the answer on your sheet of paper before 1 minute is up.

Do NOT turn to the next page.

When the minute is up the teacher will read the next page with you.

If you start with this number —————→ 12

and wish to obtain this number —————→ 20

you must use a SWITCH MACHINE.

To obtain 20 from 12 you must divide by 3 and multiply by 5.

The machine which divides by 3 and multiplies by 5 is the SWITCH MACHINE.

Exercise 7.

If you start with this number $\longrightarrow$ 15
and use a TWO-FOUR MACHINE
and then a SWITCH MACHINE
which number will you obtain?

See whether you can write the answer on your sheet of paper before 2 minutes are up.
Since you will be using two machines you will obtain two numbers. Write down both
numbers in the correct order. The second number is your answer.
Do NOT turn to the next page.

When the 2 minutes are up the teacher will read the next page with you.

If you start with this number $\longrightarrow$ 15

and use a TWO-FOUR MACHINE

you will obtain this number

If you then start with this number $\searrow \nearrow$ 30

and use a SWITCH MACHINE

you will obtain this number $\longrightarrow$ 6

15 does not divide by 4. Therefore the TWO-FOUR MACHINE multiplies it by 2.

$$15 \times 2 = 30$$

30 divides by 2 but NOT by 4. Therefore the SWITCH MACHINE divides it by 5.

$$30 \div 5 = 6$$

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Exercise 8.

If you start with this number —————→ 15
and wish to obtain this number —————→ 10
which machines will you use ?

To complete this task you will need TWO machines.

See whether you can write the names of the two machines in the correct order
on your sheet of paper before 2 minutes are up.

Do NOT turn to the next page.

When the 2 minutes are up the teacher will read the next page with you.

6

If you start with this number	—————→	15
and wish to obtain this number	—————→	10

you must use a SWITCH MACHINE
and then a TWO-FOUR MACHINE

Look at your MACHINE REMINDER sheet.

From the number 15 you can only go to 5 or to 30 in your first move.

If you go to 30 you cannot go from 30 to 10 in your second move.

If you go to 5 you can go from 5 to 10 in your second move.

Therefore you must go from 15 to 5 by using the SWITCH MACHINE and then from 5 to 10 by using the TWO-FOUR MACHINE.

Exercise 9.

If you start with this number _____→ 3

and wish to obtain this number _____→ 60

which machines would you use ?

To complete this task you will need THREE machines.

See whether you can write the names of the three machines in the correct order,
on your sheet of paper before 3 minutes are up.

Do NOT turn to the next page.

When the 3 minutes are up the teacher will read the next page with you.

If you start with this number $\longrightarrow$ 3

and wish to obtain this number $\longrightarrow$ 60

you must use a TWO-FOUR MACHINE

and then a SWITCH MACHINE

and then a TWO-FOUR MACHINE.

Look at your MACHINE REMINDER sheet.

The 3 is in the second column and the 60 is in the fourth column.

The quickest way to get from the second column to the fourth column is to go from the 6 to the 30 in one move.

Therefore you must go from the 3 to the 6 by using the TWO-FOUR MACHINE, then from the 6 to the 30 by using the SWITCH MACHINE, and then from the 30 to the 60 by using the TWO-FOUR MACHINE.

Exercise 10.

If you start with this number —————→ 1
and wish to obtain this number —————→ 5
which machines will you use ?

To complete this task you will need FOUR MACHINES.

See whether you can write the names of the four machines in the correct order
on your sheet of paper before 4 minutes are up.

Do NOT turn to the next page.

When the 4 minutes are up the teacher will read the next page with you.

If you start with this number $\longrightarrow$ 1
and wish to obtain this number $\longrightarrow$ 5
you must use a TWO-FOUR MACHINE
and then a SWITCH MACHINE
and then a TWO-FOUR MACHINE
and then a TWO-FOUR MACHINE.

Look at your MACHINE REMINDER sheet.

The 1 is in the first column and the 5 is in the third column.

The quickest way to get from the first column to the third column is to go from the 2 to the 10 in one move.

Therefore you must go from the 1 to the 2 by using a TWO-FOUR MACHINE,
then from the 2 to the 10 by using a SWITCH MACHINE,
then from the 10 to the 20 by using a TWO-FOUR MACHINE,
and then from the 20 to the 5 by using a TWO-FOUR MACHINE.

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APPENDIX G2

THE INITIAL LEARNING TEST

FOR

GAME B

QUESTIONS ON GAME B.

In each of these questions you will see a black line _____
that has nothing written on it.

Please try to write your answer on this line.

In some questions your answer will be a number, in others it will
be the name of a machine.

You may use scrap paper on which to work out your answers.

1. Your starting number is _____ → 5

If you put this number into a TWO-FOUR MACHINE
you will obtain this number

_____ → _____

2. Your starting number is _____ → 2

If you put this number into a SWITCH MACHINE
you will obtain this number _____ →

3. Your starting number is _____ → 30

If you wish to change this number into → 60

you will use this machine _____ →

4. Your starting number is _____ → 4
If you wish to change this number into → 60
you will use this machine _____ → _____

5. Your starting number is _____ → 20
If you put this number into a SWITCH MACHINE
you will obtain this number _____ → _____
If you then put this number _____ → _____
into a TWO-FOUR MACHINE
you will obtain this number _____ → _____

6. Your starting number is _____ → 60

If you put this number into a SWITCH MACHINE

you will obtain this number _____ →

If you then put this number _____ →

into a SWITCH MACHINE

you will obtain this number _____ →

7. Your starting number is _____ → 10

If you wish to change this number into → 12

You will use this machine _____ →

and then this machine _____ →

NOTE: If in questions 8, 9 and 10 you think that you can get your answer by using fewer than 3 machines then write nothing on the lines you do not need.

8. Your starting number is _____ → 1

If you wish to change this number into → 6

you will use this machine _____ →

and then this machine _____ →

and then this machine _____ →

9. Your starting number is _____ → 3

If you wish to change this number into → 10

you will use this machine _____ →

and then this machine _____ →

and then this machine _____ →

10. Your starting number is _____→ 12

If you wish to change this number into → 60

you will use this machine _____→

and then this machine _____→

and then this machine _____→

and then this machine _____→

APPENDIX G3

THE TRANSFER TEST

BETWEEN

GAME A AND GAME B

TRANSFER QUESTIONS

BETWEEN

GAME A AND GAME B

In each question you will see empty black lines _____
or empty frames , or both.

Please try to find the correct numbers or names of machines for the
empty lines, and the correct letters for the frames.

Please look carefully at your MACHINE REMINDERS for Game A and for
Game B.

To help you match up the MACHINE REMINDERS for these two Games
correctly here are three rules:

- (1) The frame **START** in Game A matches up with the number 1 in Game B.
- (2) The X MACHINE in Game A matches up with the TWO-FOUR MACHINE in Game B.
- Look at your MACHINE REMINDERS.

If in Game A you go from this frame	—————→	START
to this frame	—————→	X
by using this MACHINE	—————→	X
then in Game B you go from this number	—————→	1
to this number	—————→	2
by using this MACHINE	—————→	TWO-FOUR

(3) The Y MACHINE in Game A matches up with the SWITCH MACHINE in Game B.

Look at your MACHINE REMINDERS.

If in Game A you go from this frame —————→

START

to this frame —————→

Y

by using this MACHINE —————→

Y

then in Game B you go from this number —————→

1

to this number —————→

3

by using this MACHINE —————→

SWITCH

Now try to answer the questions.

1. If in Game A you start with this frame _____

X

and use this machine _____

X

to go to this frame _____

then in Game B you start with this number _____

2

and use this machine _____

TWO-FOUR

to go to this number _____

2. If in Game B you start with this number $\longrightarrow$ 15

and use this machine $\longrightarrow$ SWITCH

to go to this number $\longrightarrow$ _____

then in Game A you start with this frame $\longrightarrow$ $\bigcirc X^2 YX$

and use this machine $\longrightarrow$ Y

to go to this frame $\longrightarrow$ $\bigcirc$

3. If in Game A you start with this frame	→	<u>Y</u>
and then this machine	→	X
and then this machine	→	X
to go to this frame	→	<u> </u>
then in Game B you start with this number	→	3
and use this machine	→	TWO-FOUR
and then this machine	→	TWO-FOUR
to go to this number	→	<u> </u>

4. If in Game B you start with this number → 5

and use this machine → SWITCH

and then this machine → TWO-FOUR

to go to this number → _____

then in Game A you start with this frame → XYX^2

and use this machine → Y

and then this machine → X

to go to this frame → 

5. If in Game A you go from this frame



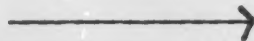
x^2

to this frame

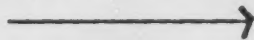


Y

by using this machine



and then this machine



then in Game B you go from this number



4

to this number



by using this machine



and then this machine



6. If in Game B you go from this number $\longrightarrow$ 60
to this number $\longrightarrow$ 1
by using this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____
then in Game A you go from this frame $\longrightarrow$ x^2y
to this frame $\longrightarrow$ _____
by using this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____

7. If in Game A you go from this frame —————>

YX^2

to this frame —————>

$\begin{matrix} 2 \\ X \ YX \end{matrix}$

by using this machine —————>

and then this machine —————>

and then this machine —————>

then in Game B you go from this number —————>

12

to this number —————>

by using this machine —————>

and then this machine —————>

and then this machine —————>

8. If in Game B you go from this number —————>

30

to this number —————>

3

by using this machine —————>

and then this machine —————>

and then this machine —————>

then in Game A you go from this frame —————>

YXY

to this frame —————>

by using this machine —————>

and then this machine —————>

and then this machine —————>

9. If in Game A you go from this frame —————>

to this frame —————>

by using this machine —————>

and then this machine —————>

and then this machine —————>

and then this machine —————>

then in Game B you go from this number —————>

to this number —————>

by using this machine —————>

and then this machine —————>

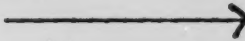
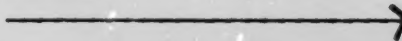
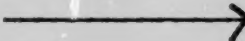
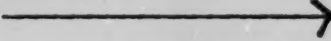
and then this machine —————>

and then this machine —————>

YX

XY

6

10. If in Game B you go from this number  10
to this number  30
by using this machine  _____
and then this machine  _____
and then this machine  _____
and then this machine  _____
then in Game A you go from this frame  
to this frame  
by using this machine  _____
and then this machine  _____
and then this machine  _____
and then this machine  _____

APPENDIX G4.

THE TRANSFER TEST

BETWEEN

GAME B AND GAME C

TRANSFER QUESTIONS
BETWEEN
GAME B AND GAME C

In each question you will see either black lines _____

or empty frames

--	--	--	--

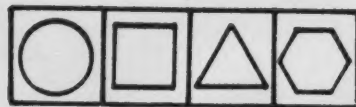
or both.

Please try to find the correct shapes for the empty frames,
and the correct numbers or names of machines for the black lines.

Please look carefully at your MACHINE REMINDERS for Game B and Game C
as you try to answer each question.

To help you match up the MACHINE REMINDERS for these two Games
correctly here are three rules:

(1) The frame



in Game C matches up with

the number in Game B.

(2) The FIRST MACHINE in Game C matches up with the TWO-FOUR MACHINE in Game B.

Look at your MACHINE REMINDERS.

If in Game C you go from this frame →



to this frame →



by using the FIRST MACHINE

then in Game B you go from this number → 1

to this number → 2

by using the TWO-FOUR MACHINE.

(3) The SWITCH MACHINE in Game C matches up with the SWITCH MACHINE in Game B.

Look at your MACHINE REMINDERS.

If in Game C you go from this frame —————→



to this frame —————→



by using the SWITCH MACHINE

then in GAME B you go from this number —————→

1

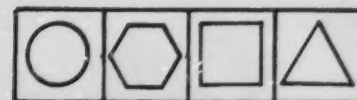
to this number —————→

3

by using the SWITCH MACHINE.

Now try to answer the questions.

1. If in Game C you start with this frame —————→



and use this machine —————→

FIRST

to go to this frame —————→



then in Game B you start with this number —————→

2

and use this machine —————→

TWO-FOUR

to go to this number —————→

2. If in Game B you start with this number —————>

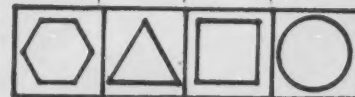
15

and use this machine —————>

SWITCH

to go to this number —————>

then in Game C you start with this frame —————>



and use this machine —————>

SWITCH

to go to this frame —————>



3. If in Game C you start with this frame —→



and use this machine —→

FIRST

and then this machine —→

FIRST

to go to this frame —→



then in Game B you start with this number —→

3

and use this machine —→

TWO-FOUR

and then this machine —→

TWO-FOUR

to go to this number —→

4. If in Game B you start with this number —————>

5

and use this machine —————>

SWITCH

and then use this machine —————>

TWO-FOUR

to go to this number —————>

then in Game C you start with this frame —————>



and use this machine —————>

SWITCH

and then this machine —————>

FIRST

to go to this frame —————>



5. If in Game C you go from this frame →



to this frame →



by using this machine →

and then this machine →

then in Game B you go from this number →

4

to this number →

by using this machine →

and then this machine →

6. If in Game B you go from this number —————→ 60

to this number —————→ 1

by using this machine —————→ _____

and then this machine —————→ _____

then in Game C you go from this frame —————→



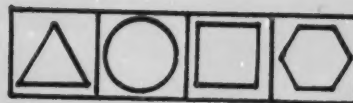
to this frame —————→



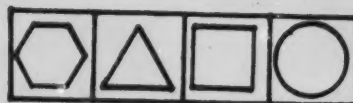
by using this machine —————→ _____

and then this machine —————→ _____

7. If in Game C you go from this frame —→



to this frame —→



by using this machine —→

and then this machine —→

and then this machine —→

then in Game B you go from this number —→

12

to this number —→

by using this machine —→

and then this machine —→

and then this machine —→

8. If in Game B you go from this number —————>

30

to this number —————>

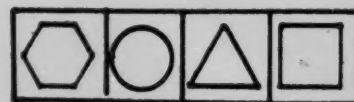
3

by using this machine —————>

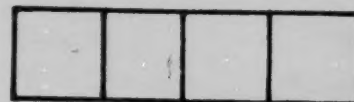
and then this machine —————>

and then this machine —————>

then in Game C you go from this frame —————>



to this frame —————>



by using this machine —————>

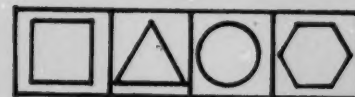
and then this machine —————>

and then this machine —————>

9. If in Game C you go from this frame —————>



to this frame —————>



by using this machine —————>

and then this machine —————>

and then this machine —————>

and then this machine —————>

then in Game B you go from this number —————>

6

to this number —————>

by using this machine —————>

and then this machine —————>

and then this machine —————>

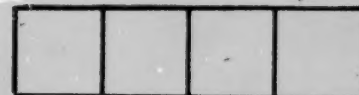
and then this machine —————>

10. If in Game B you go from this number $\longrightarrow$ 10
to this number $\longrightarrow$ 30
by using this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____

then in Game C you go from this frame $\longrightarrow$



to this frame $\longrightarrow$



by using this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____

- 00 -

APPENDIX G5.

THE TRANSFER TEST
BETWEEN
GAME B AND GAME D

419

419

TRANSFER QUESTIONS

BETWEEN

GAME B AND GAME D

In each question you will see empty black lines _____

Please try to find the correct sentences, numbers, or names of machines for these black lines.

Please look carefully at your MACHINE REMINDERS for Game B and for Game D as you try to answer each question.

To help you match up the MACHINE REMINDERS for these two Games correctly here are three rules:

- (1) The sentence JOHN IS EATING in Game D matches up with the number I in Game B.
- (2) The TIME MACHINE in Game D matches up with the TWO-FOUR MACHINE in Game B.

Look at your MACHINE REMINDERS.

If in Game D you go from this sentence	————→	JOHN IS EATING
to this sentence	————→	JOHN WILL BE EATING
by using this MACHINE	————→	TIME
then in Game B you go from this number	————→	1
to this number	————→	2
by using this MACHINE	————→	TWO-FOUR

(3) The SWITCH MACHINE in GAME D matches up with the SWITCH MACHINE in Game B.

Look at your MACHINE REMINDERS.

If in Game D you go from this sentence	————→	JOHN IS EATING
to this sentence	————→	PAUL IS EATING
by using this MACHINE	————→	SWITCH
then in Game B you go from this number	————→	1
to this number	————→	3
by using this MACHINE	————→	SWITCH

Now try to answer the questions.

1. If in Game D you start with this sentence —————→ JOHN WILL BE EATING
- and use this machine —————→ TIME
- to go to this sentence —————→ _____
- then in Game B you start with this number —————→ 2
- and use this machine —————→ TWO-FOUR
- to go to this number —————→ _____

2. If in Game B you start with this number → 15
and use this machine → SWITCH
to go to this number → _____
then in Game D you start with this sentence → PAUL IS DRINKING
and use this machine → SWITCH
to go to this sentence → _____

3. If in Game D you start with this sentence	→	PAUL IS EATING
and use this machine	→	TIME
and then this machine	→	TIME
to go to this sentence	→	_____
then in Game B you start with this number	→	3
and use this machine	→	TWO-FOUR
and then this machine	→	TWO-FOUR
to go to this number	→	_____

4. If in Game B you start with this number → 5
and use this machine → SWITCH
and then this machine → TWO-FOUR
to go to this number → _____
then in Game D you start with this sentence → JOHN IS DRINKING
and use this machine → SWITCH
and then this machine → TIME
to go to this sentence → _____

5.	If in Game D you go from this sentence	→	JOHN WAS EATING
	to this sentence	→	PAUL IS EATING
	by using this machine	→	_____
	and then this machine	→	_____
	then in Game B you go from this number	→	<u>4</u>
	to this number	→	_____
	by using this machine	→	_____
	and then this machine	→	_____

6. If in Game B you go from this number → 60
to this number → 1
by using this machine → _____
and then this machine → _____
then in Game D you go from this sentence → PAUL WAS DRINKING
to this sentence → _____
by using this machine → _____
and then this machine → _____

7.	If in Game D you go from this sentence	→	PAUL WAS EATING
	to this sentence	→	PAUL IS DRINKING
	by using this machine	→	_____
	and then this machine	→	_____
	and then this machine	→	_____
	then in Game B you go from this number	→	12 _____
	to this number	→	_____
	by using this machine	→	_____
	and then this machine	→	_____
	and then this machine	→	_____

8. If in Game B you go from this number



30

to this number

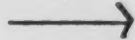


3

by using this machine



and then this machine



and then this machine



then in Game D you go from this sentence



PAUL WILL BE DRINKING

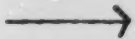
to this sentence



by using this machine



and then this machine



and then this machine



9.	If in Game D you go from this sentence —————→	PAUL WILL BE EATING
	to this sentence —————→	JOHN WILL BE DRINKING
	by using this machine —————→	_____
	and then this machine —————→	_____
	and then this machine —————→	_____
	and then this machine —————→	_____
	then in Game B you go from this number —————→	6
	to this number —————→	_____
	by using this machine —————→	_____
	and then this machine —————→	_____
	and then this machine —————→	_____
	and then this machine —————→	_____

10. If in Game B you go from this number —→

10

to this number —→

30

by using this machine —→

and then this machine —→

and then this machine —→

and then this machine —→

then in Game D you go from this sentence →

JOHN WILL BE DRINKING

to this sentence →

by using this machine —→

and then this machine —→

and then this machine —→

and then this machine —→

APPENDIX G6.

THE TRANSFER TEST

BETWEEN

GAME B AND GAME F

TRANSFER QUESTIONS

BETWEEN

GAME B AND GAME F

In each question you will see empty black lines

Please try to find the correct names of the machines
for the empty black lines.

Please look carefully at your MACHINE REMINDERS for Game B and
Game F as you try to answer each question.

To help you match up the MACHINE REMINDERS for these two Games
correctly, here are four rules:

- (1) The numbers in Game B are the same as the numbers in Game F.
- (2) The TWO-FOUR MACHINE in Game B is also the TWO-FOUR MACHINE in Game F.

Look at your MACHINE REMINDERS.

If in Game B you go from this number —————→ 1
to this number —————→ 2

by using the TWO-FOUR MACHINE

then in Game F you go from this number —————→ 1
to this number —————→ 2

by using the TWO-FOUR MACHINE

- (3) The SWITCH MACHINE in Game B does NOT match up with the SPECIAL MACHINE in Game F.

Look at your MACHINE REMINDERS.

If in Game B you go from this number —————→ 1
to this number —————→ 3

by using the SWITCH MACHINE

then in Game F you go from this number —————→ 1
to this number —————→ 3

by using the TWO-FOUR MACHINE
and then the SPECIAL MACHINE.

- (4) The SPECIAL MACHINE in Game F does NOT match up with the SWITCH MACHINE in Game B.

Look at your MACHINE REMINDERS.

If in Game F you go from this number —————→ 1

to this number —————→ 60

by using the SPECIAL MACHINE

then in Game B you go from this number —————→ 1

to this number —————→ 60

by using the TWO-FOUR MACHINE
and then the TWO-FOUR MACHINE
and then the SWITCH MACHINE.

So you see that the number of machines you have to use to go from one number to another is not always the same in the two Games.

Now try to answer the questions

1. If in Game B you go from this number → 2
to this number → 10
by using this machine → _____
then in Game F you go from this number → 2
to this number → 10
by using this machine → TWO-FOUR
and then this machine → _____

2. If in Game F you go from this number	—————>	15
to this number	—————>	4
by using this machine	—————>	_____
then in Game B you go from this number	—————>	15
to this number	—————>	4
by using this machine	—————>	TWO-FOUR
and then this machine	—————>	TWO-FOUR
and this machine	—————>	_____

3. If in Game B you go from this number —————→ 3
to this number —————→ 2
by using this machine —————→ _____
and then this machine —————→ _____

then in Game F you go from this number —————→ 3
to this number —————→ 2
by using this machine —————→ _____
and then this machine —————→ _____

4. If in Game F you go from this number $\longrightarrow$ 5
to this number $\longrightarrow$ 2
by using this machine $\longrightarrow$ TWO-FOUR
and then this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____

then in Game B you go from this number $\longrightarrow$ 5
to this number $\longrightarrow$ 2
by using this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____

5.	If in Game B you go from this number	—————>	4
	to this number	—————>	30
	by using this machine	—————>	SWITCH
	and then this machine	—————>	_____
	and then this machine	—————>	_____
	then in Game F you go from this number	—————>	4
	to this number	—————>	30
	by using this machine	—————>	SPECIAL
	and then this machine	—————>	_____
	and then this machine	—————>	_____

6. Although your starting and your final number are the same in this question you can use machines which will take you away from this number and then back to it.

If in Game F you go from this number	—————>	60
to this number	—————>	60
by using this machine	—————>	SPECIAL
and then this machine	—————>	_____
and then this machine	—————>	_____
then in Game B you go from this number	—————>	60
to this number	—————>	60
by using this machine	—————>	_____
and then this machine	—————>	_____

7.	If in Game B you go from this number	—————→	12
	to this number	—————→	2
	by using this machine	—————→	_____
	and then this machine	—————→	_____
	and then this machine	—————→	_____
	then in Game F you go from this number	—————→	12
	to this number	—————→	2
	by using this machine	—————→	_____
	and then this machine	—————→	_____
	and then this machine	—————→	_____

8.	If in Game F you go from this number	—————→	30
	to this number	—————→	20
	using this machine	—————→	_____
	and then this machine	—————→	_____
	and then this machine	—————→	_____
	then in Game B you go from this number	—————→	30
	to this number	—————→	20
	by using this machine	—————→	_____
	and then this machine	—————→	_____
	and then this machine	—————→	_____

9. PLEASE NOTE: If in this question you find that you can go from the first number to the second number by using fewer than four machines, then write nothing on the empty lines you do not need.

cont'd.

9. If in Game B you go from this number —————→ 10

to this number —————→ 30

by using this machine —————→ _____

and then this machine —————→ _____

and then this machine —————→ _____

and then this machine —————→ _____

then in Game F you go from this number —————→ 10

to this number —————→ 30

by using this machine —————→ _____

and then this machine —————→ _____

and then this machine —————→ _____

and then this machine —————→ _____

10. PLEASE NOTE: If in this question you find that you can go from the first number to the second number by using fewer than four machines, then write nothing on the empty lines you do not need.

cont'd

10.	If in Game F you go from this number	—————→	6
	to this number	—————→	10
	by using this machine	—————→	_____
	and then this machine	—————→	_____
	and then this machine	—————→	_____
	and then this machine	—————→	_____
	then in Game B you go from this number	—————→	6
	to this number	—————→	10
	by using this machine	—————→	_____
	and then this machine	—————→	_____
	and then this machine	—————→	_____
	and then this machine	—————→	_____

APPENDIX G7.

THE TRANSFER TEST

BETWEEN

GAME B AND GAME G

TRANSFER QUESTIONS

BETWEEN

GAME B AND GAME G

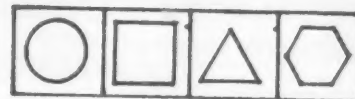
In each question you will see empty black lines

Please try to find the correct names of the machines
for the empty black lines.

Please look carefully at your MACHINE REMINDERS for Game B and
Game G as you try to answer each question.

To help you match up your MACHINE REMINDERS for these two Games
correctly, here are four rules:

(1) The number 1 in Game B matches up with the frame



in Game G.

(2) The TWO-FOUR MACHINE in Game B matches up with the FIRST MACHINE in Game G.

Look at your MACHINE REMINDERS.

If in Game B you go from this number —————→

1

to this number —————→

2

by using the TWO-FOUR MACHINE

then in Game G you go from this frame —————→



to this frame —————→



by using the FIRST MACHINE.

(3) The SWITCH MACHINE in Game B does NOT match up with the SECOND MACHINE in Game G.

Look at your MACHINE REMINDERS.

If in Game B you go from this number —————→ 1

to this number —————→ 3

by using the SWITCH MACHINE

then in Game G you go from this frame —————→



to this frame —————→



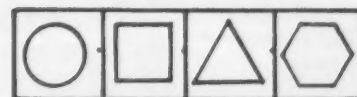
by using the FIRST MACHINE

and then the SECOND MACHINE.

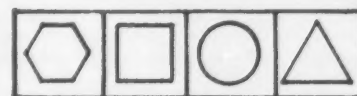
- (4) The SECOND MACHINE in Game G does NOT match up with the SWITCH MACHINE in Game B.

Look at your MACHINE REMINDERS.

If in Game G you go from this frame —————→



to this frame —————→



by using the SECOND MACHINE

then in Game B you go from this number —————→

1

to this number —————→

60

by using the TWO-FOUR MACHINE

and then the TWO-FOUR MACHINE

and then the SWITCH MACHINE.

So you see that you do not always use the same number of machines in the two Games when you answer a question.

Now try to answer the questions

1. If in Game B you go from this number $\longrightarrow$ 2
to this number $\longrightarrow$ 10
by using this machine $\longrightarrow$

then in Game G you go from this frame $\longrightarrow$



to this frame $\longrightarrow$



by using this machine $\longrightarrow$

FIRST

and then this machine $\longrightarrow$

2. If in Game G you go from this frame



to this frame



by using this machine



then in Game B you go from this number



15

to this number



4

by using this machine



TWO-FOUR

and then this machine



TWO-FOUR

and then this machine



If in Game B you go from this number $\longrightarrow$ 3
 to this number $\longrightarrow$ 2
 by using this machine $\longrightarrow$ _____
 and then this machine $\longrightarrow$ _____
 then in Game G you go from this frame $\longrightarrow$

△	⬡	○	□
---	---	---	---

 to this frame $\longrightarrow$

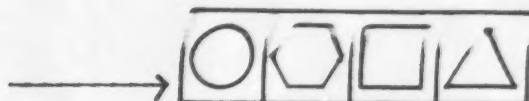
○	⬡	□	△
---	---	---	---

 by using this machine $\longrightarrow$ _____
 and then this machine $\longrightarrow$ _____

4. If in Game G you go from this frame



to this frame



by using this machine



FIRST

and then this machine



and then this machine



then in Game B you go from this number



5

to this number



2

by using this machine



and then this machine



If in Game B you go from this number



4

to this number



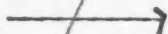
30

by using this machine



SWITCH

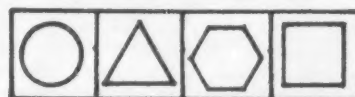
and then this machine



and then this machine



then in Game G you go from this frame



to this frame



by using this machine



SECOND

and then this machine



and then this machine



Although your starting and your final frame and also your starting and your final number are the same in this question you can use machines which will take you away and then back to them.

If in Game G you go from this frame $\longrightarrow$ 

to this frame $\longrightarrow$ 

by using this machine $\longrightarrow$ SECOND

and then this machine $\longrightarrow$ _____

and then this machine $\longrightarrow$ _____

then in Game B you go from this number $\longrightarrow$ _____

60

to this number $\longrightarrow$ _____

60

by using this machine $\longrightarrow$ _____

and then this machine $\longrightarrow$ _____

If in Game B you go from this number —————> 12

to this number —————> 2

by using this machine —————> _____

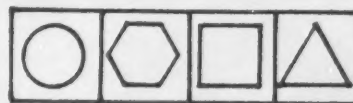
and then this machine —————> _____

and then this machine —————> _____

then in Game G you go from this frame —————>



to this frame —————>



by using this machine —————> _____

and then this machine —————> _____

and then this machine —————> _____

If in Game G you go from this frame



to this frame



by using this machine



and then this machine



and then this machine



then in Game B you go from this number



30

to this number



20

by using this machine



and then this machine



and then this machine

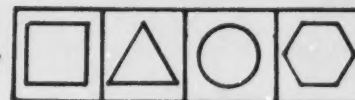


PLEASE NOTE: If in this question you find that you can go from the first number to the second number, or from the first frame to the second frame, by using fewer than four machines, then write nothing on the empty lines you do not need.

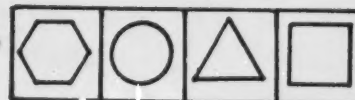
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9. If in Game B you go from this number $\longrightarrow$ 10
to this number $\longrightarrow$ 30
by using this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____

then in Game G you go from this frame $\longrightarrow$



to this frame $\longrightarrow$



by using this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____
and then this machine $\longrightarrow$ _____

PLEASE NOTE: If in this question you find that you can go from the first number to the second number, or from the first frame to the second frame, by using fewer than four machines, then write nothing on the empty lines you do not need.

cont'd.

10. If in Game G you go from this frame



to this frame



by using this machine



and then this machine



and then this machine



and then this machine



then in Game B you go from this number



6

to this number



10

by using this machine



and then this machine



and then this machine



and then this machine